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Self-similar gravity currents with variable inflow revisited: plane currents

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We use shallow-water theory to study the self-similar gravity currents that describe the intrusion of a heavy fluid below a lighter ambient fluid. We consider in detail the case of currents with planar symmetry produced by a source with variable inflow, such that the volume of the intruding fluid varies in time according to a power law of the type t^{α}. The resistance of the ambient fluid is taken into account by a boundary condition of the von Kármán type, that depends on a parameter β that is a function of the density ratio of the fluids. The flow is characterized by β, α, and the Froude number $\mathscr{F}_0$ near the source. We find four kinds of self-similar solutions: subcritical continuous solutions (Type I), continuous solutions with a supercritical–subcritical transition (Type II), discontinuous solutions (Type III) that have a hydraulic jump, and discontinuous solutions having hydraulic jumps and a subcritical–supercritical transition (Type IV). The current is always subcritical near the front, but near the source it is subcritical ($\mathscr{F}_0 < 1$) for Type I currents, and supercritical ($\mathscr{F}_0 > 1$) for Types II, III, and IV. Type I solutions have already been found by other authors, but Type II, III, and IV currents are novel. We find the intervals of parameters for which these solutions exist, and discuss their properties. For constant-volume currents one obtains Type I solutions for any β that, when $\beta > 2$, have a 'dry' region near the origin. For steady inflow one finds Type I currents for $0 < \beta < \infty$ and Type II and III currents for any β, if $\mathscr{F}_0$ is sufficiently large.

1. Introduction

Gravity currents in fluids are frequent in nature and manmade situations, and are important for theoretical and practical reasons (Simpson 1982). Various studies of gravity currents (Simpson & Britter 1979; Huppert & Simpson 1980; Rottman & Simpson 1983, 1984) deal with the flows produced by the instantaneous release of a given volume of a heavy fluid within an ambient fluid of lower density, a problem that is relevant for the assessment of the risks associated with the accidental release of toxic or flammable gases due to the rupture of storage tanks or pipelines. Variable inflow currents are also of interest in this context, to describe the effect of leaks of punctures through which the fluid is released over a period of time. A series of experiments on plane currents with variable inflow was performed by Maxworthy (1983).

If viscosity is negligible, the balance of gravity and inertial forces governs the flow, and the shallow-water equations may be used. This description is adequate if the length of the current greatly exceeds its depth, which may not be the case at the beginning of the phenomenon. In this situation the governing equations admit a family of self-similar solutions that represent the intermediate asymptotics of a variety of initial- and boundary-value problems. Self-similar inertial gravity currents have been studied theoretically by Britter (1979), who considered the spreading of a constant volume of liquid, and experimentally by Huppert & Simpson (1980).

Grundy & Rottman (1985, 1986, from now on designated GR) investigated plane and axisymmetric currents whose volume varies with time that intrude in an ambient fluid. To take into account the resistance of the latter they used a boundary condition of the von Kármán type at the current front. This condition depends on a parameter β that is a function of the density ratio of the fluids (von Kármán 1940). Yet the results of GR are not entirely satisfactory: for plane symmetry no solutions were found for several β ranges (including $\beta \rightarrow \infty$, that corresponds to an ambient fluid of vanishing density) and for other β they find a puzzling multiplicity of solutions; for axial symmetry no self-similar solutions were found at all. For plane symmetry, GR suggested that the 'missing' solutions could be found by considering currents with a hydraulic jump, yet did not examine this possibility in any detail. The puzzles of the non-unique solutions for plane symmetry, and their non-existence for axial geometry remain. Considering the importance of gravity currents, more work is necessary, as GR state in their paper, to clarify these matters fully.

Here we reinvestigate the self-similar inertial gravity currents produced by the intrusion of a heavy fluid beneath a lighter ambient fluid that rests on a horizontal rigid surface. The intruding fluid issues from a source at the origin of coordinates, and its volume varies with time according to a power law of the type t^{α}. We consider in detail currents with plane geometry, and show that there are unique self-similar solutions for any β; we discuss their properties and physical interpretation. Thus we solve the puzzles and ambiguities of the previous work and obtain a fully satisfactory understanding of the plane currents. We leave for future work the case of axial geometry.

The paper is divided as follows: in §2 we briefly review the governing equations, the boundary conditions, the phase-plane formalism (Sedov 1959; Courant & Friedrichs 1948), and the limits of validity of the theory (Huppert 1982). In §3 we consider the different types of self-similar solutions and their construction; for this purpose it is essential to characterize adequately the flow by specifying, besides β and α, the Froude number $\mathscr{F}_0$ of the current near the source. In §4 we investigate the existence and uniqueness of the solutions and their properties, we show several solutions as examples of the different kinds of currents that can occur, and we discuss the intervals of β, α and $\mathscr{F}_0$ in which they appear. In §5 we consider special analytic solutions, including the important cases $\alpha = 0, 1$. Section 6 presents the conclusions.

Our main result is that there are four kinds of self-similar solutions: subcritical continuous solutions (Type I), continuous solutions with a supercritical–subcritical transition (Type II), discontinuous solutions (Type III) with a hydraulic jump, and discontinuous solutions having two hydraulic jumps and a subcritical–supercritical transition (Type IV). The current is always subcritical near the front, but near the source it is subcritical ($\mathscr{F}_0 < 1$) for Type I currents, and supercritical ($\mathscr{F}_0 > 1$) for Types II, III, and IV. Type I solutions were found by GR, but Type II, III, and IV currents are novel. In Type II currents the supercritical–subcritical transition occurs without discontinuity. Type III currents have a single hydraulic jump. In Type IV currents the transition occurs in three steps: first, there is a hydraulic jump connecting the source part of the current with an intermediate subcritical flow region; second, the intermediate flow has a continuous subcritical–supercritical transition, passing into an intermediate supercritical region; finally, there is a second hydraulic jump connecting the intermediate supercritical region with the front region. When $\alpha = 0$ (constant-volume currents) one obtains Type I solutions for any β. When $\alpha = 1$ (steady inflow) one obtains Type I, II and III currents. In general for any β it is always possible to find a family of self-similar solutions that represent currents produced by sources with different combinations of α and $\mathscr{F}_0$. When the current is subcritical everywhere (Type I), β and $\mathscr{F}_0$ must be compatible, so that $\mathscr{F}_0 = \mathscr{F}_0(\beta) < 1$. Otherwise (Types II, III, IV) for any β it is possible to choose $\mathscr{F}_0$ independently.

2. Theory

We consider self-similar gravity currents in which an incompressible fluid (density ρ) intrudes beneath a lighter ambient fluid (density ρ_a) that rests on a horizontal surface. We neglect friction between the fluids and the bottom. Under certain conditions to be discussed later, the balance between the inertia of the fluid and the forces due to gravity (g is acceleration due to gravity) and buoyancy govern the phenomenon, the effects of viscosity being negligible. We shall assume that the length of the current is much larger than its depth, so that the vertical accelerations can be neglected, and the pressure is hydrostatic (Lamb 1945). We shall also assume that the depth of the ambient fluid is much larger than the thickness of the current. Then the flow can be described by the velocity u and the depth h of the intruding fluid. If the current has Cartesian or axial symmetry, u, h will depend on time (t) and on a single spatial coordinate x; for Cartesian symmetry x is the distance from a linear source, while for axial symmetry it is the distance from a point source. In this paper we shall be concerned with plane currents, but in this Section we shall present formulae that include the axially symmetric case, since it does not add complications, and will help future work.

2.1. *Governing equations and boundary conditions*

With the above assumptions, the momentum and continuity equations can be written as (Penney & Thornhill 1952)

$$u_t + uu_x + g h_x = 0, \quad h_t + x^{-n}(x^n u h)_x = 0. \tag{1}$$

Here the geometrical index n takes the value 0 for Cartesian and 1 for axial geometry, the suffixes denote derivatives, and

$$g = g(\rho - \rho_a)/\rho. \tag{2}$$

Equations (1) are similar to those of the shallow-water theory (see for example Landau & Lifschitz 1959) with g replaced by the 'reduced' gravity g.

If the current has a hydraulic jump h and u will be discontinuous. Hydraulic jumps in the interface between two superimposed fluids are called 'internal' jumps, and have been studied by Yih & Guha (1955). In our coordinate system the jump is moving with a velocity c, and the jump conditions (for the case in which the depth of the ambient fluid is much larger than that of the intruding one, see Yih 1965) are

$$u' - c = 2(u-c)/\phi(\mathscr{F}), \quad h' = \tfrac{1}{2} h \phi(\mathscr{F}), \tag{3}$$

in which the Froude number $\mathscr{F}$ is given by

$$\mathscr{F} = (u-c)/(g h)^{\frac{1}{2}}, \quad \phi(\mathscr{F}) = (1+8\mathscr{F}^2)^{\frac{1}{2}} - 1, \tag{4}$$

and we denote with primes the variables after the jump. Equations (3), (4) are identical to the conditions for an ordinary hydraulic jump, with g replaced by g.

We are considering currents produced by a source with variable inflow located at $x = 0$; the volume of the current (volume per unit width for $n = 0$, total volume for $n = 1$) is given by

$$\mathscr{Q}(t) = q_\alpha t^\alpha, \quad \alpha \geqslant 0, \tag{5}$$

within an arbitrary constant. The boundary condition at the source is

$$\lim_{x\to 0}[(2\pi x)^n u h] = \alpha q_\alpha t^{\alpha-1}. \tag{6}$$

It is important to realize that condition (6) does not specify completely the characteristics of the source. To see this point, it is helpful to discuss briefly how a flow that satisfies (6) can be actually set up, without entering into practical details. For $\alpha = 1$, it is clear that any source that generates a flow with u, h constant is adequate; this can be achieved, for example, by draining a reservoir in which the liquid is kept at a constant depth through a slit opening of width h (in the vertical direction) at the bottom of one of its sides. For $\alpha \neq 1$ the same type of source might be used, but now the depth of the liquid in the reservoir, or the width of the slit, or both, must be varied with time so as to satisfy (6).

Clearly in an experiment there is an infinite number of ways to set up a source that gives the flow (6) since an infinite number of choices of $h(x=0)$ and $u(x=0)$ are possible. In other words, we need some additional condition to determine uniquely the properties of the source, otherwise we cannot expect to have unique solutions, except under special conditions†. Many additional conditions can be imagined, for example one might require that $h(x=0)$ (or $u(x=0)$) has a given constant value. These possibilities are legitimate, but will spoil the self-similarity of the problem by introducing an additional constant dimensional parameter. The only way to preserve self-similarity is that the new parameter arising from the additional condition be dimensionless. Of course this requires a source with special properties, but this is a price we must pay to achieve self-similarity. The only additional dimensionless parameter that can be used to specify the source is the 'source Froude number' $\mathscr{F}_0$ given by

$$\mathscr{F}_0 = \lim_{x\to 0}\mathscr{F} = \lim_{x\to 0}[u/(gh)^{\frac{1}{2}}]. \tag{7}$$

The condition $\mathscr{F}_0 = \text{constant}$ requires $u(x=0) = u_0 t^p$, $h(x=0) = h_0 t^{2p}$ with $p = \frac{1}{3}(\alpha-1) = \delta - 1$; this may be complicated from an experimental point of view, but probably not much more than satisfying only the condition (6). The choice of (7) as a second source condition has an additional advantage: it distinguishes the source that produce a subcritical flow ($\mathscr{F}_0 < 1$) from those that produce a supercritical current ($\mathscr{F}_0 > 1$), which lead to very different physical situations, and as we shall see later yield solutions of a different character. GR did not include the source condition (7) which, as we show, resolves the non-uniqueness question associated with their solutions for $\mathscr{F}_0 > 1$.

To complete the discussion of the source boundary conditions we mention that there is an equivalent way of expressing (6), that can be useful in some circumstances. In fact $\mathscr{Q}$ satisfies the integral condition

$$\mathscr{Q} = \int_0^{x_f(t)} (2\pi x)^n h(x,t)\,\mathrm{d}x, \tag{8}$$

in which $x_f(t)$ denotes the position of the front. The condition (8) is not independent of (6), but is a consequence of it and of the mass conservation equation (and of the jump conditions), so that it is satisfied automatically for any current satisfying (6).

† In fact, as it will be shown later, this is precisely what happens when the front boundary condition forces the current to be entirely subcritical, in which case a unique solution is obtained regardless of the properties of the source, which has been choked (i.e. $h(x=0)$ and $u(x=0)$ become independent of the details of the source, with the exception of α).

The boundary condition at the front must describe the resistance of the ambient fluid to the advance of the intruding fluid. We use the condition

$$\beta^2 g\hbar(x_f, t) = u^2(x_f, t) = \dot{x}_f^2, \tag{9}$$

in which $\dot{x}_f$ is the front velocity, and β is a constant dimensionless parameter. Equation (9) expresses a quasi-stationary balance between the driving forces (pressure gradient and buoyancy) and the drag due to the acceleration of the ambient fluid around the front (that produces a drag proportional to u^2; a detailed discussion of this subject can be found in the review article of Simpson 1982). We can write this balance as

$$\Delta p \approx \tfrac{1}{2}(\rho-\rho_a)\, g\hbar(x_f, t) \approx \tfrac{1}{2}\rho_a u^2(x_f, t), \tag{10}$$

and we obtain

$$\mathscr{B}\mathscr{G}\hbar(x_f, t) \approx u^2(x_f, t), \tag{11}$$

in which $\mathscr{G} = g(\rho-\rho_a)/\rho_a$, and $\mathscr{B}$ is a dimensionless parameter of the order of unity (von Kármán 1940). Since $\mathscr{G} \neq g$ it is convenient to rewrite (11) in terms of the latter; to this end we define $\beta^2 = \mathscr{B}\rho/\rho_a$ so that (11) reduces to (9). The parameter β could be determined experimentally, and if the densities of the two fluids are similar it will be close to unity (see for example GR, notice that in this work there is a misprint in the definition of g); the case of a gravity current in vacuum ($\rho_a = 0$) corresponds to $\beta \to \infty$, i.e. to $\hbar(x_f, t) \to 0$.

2.2. *Self-similar gravity currents*

The formal analogy between shallow-water theory and the dynamics of an ideal gas with adiabatic exponent $\gamma = 2$ (see for example Landau & Lifschitz 1959) allows one to adapt to the present problem the phase-plane formalism of gas dynamics (Sedov 1959; Courant & Friedrichs 1948), which is useful for a systematic derivation of the self-similarities. The analogy is not complete for discontinuous solutions, since the hydraulic jump conditions are different from the Rankine–Hugoniot conditions for gas shocks.

The gravity currents with variable inflow given by (5) are self-similar. To see this more clearly, lets us write $h = g\hbar$, $q_\alpha = g\mathscr{q}_\alpha$. We obtain from (1)

$$u_t + uu_x + h_x = 0, \quad h_t + x^{-n}(x^n uh)_x = 0; \tag{12}$$

the jump conditions (3) are

$$u' - c = 2(u-c)/\phi(\mathscr{F}), \quad h' = \tfrac{1}{2}h\,\phi(\mathscr{F}), \tag{13}$$

with $\mathscr{F} = (u-c)/h^{\frac{1}{2}}$. The source boundary conditions (6), (7) are

$$\lim_{x\to 0}\,[(2\pi x)^n\, uh] = \alpha q_\alpha t^{\alpha-1}, \quad \lim_{x\to 0} uh^{-\frac{1}{2}} = \mathscr{F}_0, \tag{14}$$

and the condition at the front (9) is

$$\beta^2 h(x_f, t) = u^2(x_f, t) = \dot{x}_f^2. \tag{15}$$

In (12) no constant dimensional parameter appears (the role of g is simply to give a depth scale). From the variables and characteristic parameters of the problem one can form five dimensionless combinations, that can be taken as (see for example Sedov 1959; Zel'dovich & Raizer 1968):

$$\Pi = ut/x, \quad \Phi = ht^2/x^2, \quad \xi = x/(bt^\delta), \quad \beta, \quad \mathscr{F}_0, \tag{16}$$

with† $$\delta = 1/\mu = (2+\alpha)/(3+n), \quad b = q_\alpha^{1/(3+n)}. \tag{17}$$

We must have $\Pi = \Pi(\beta, \mathscr{F}_0, \xi)$, $\Phi = \Phi(\beta, \mathscr{F}_0, \xi)$, so that

$$u = (x/t)\,\Pi(\beta, \mathscr{F}_0, \xi), \quad h = (x/t)^2\,\Phi(\beta, \mathscr{F}_0, \xi), \tag{18}$$

which means that the flow is self-similar in the variable ξ. The self-similarity appears because there is in the problem only one constant dimensional parameter (b), so that only one independent dimensionless variable can be formed from x, t.

Notice that since $\alpha \geqslant 0$ one has $\delta \geqslant 2/(3+n)$. On the other hand the validity of the present theory (which we shall discuss below) requires $\alpha < 4+2n$, so that $\delta < 2$. Then μ is restricted to the interval $\frac{1}{2} < \mu \leqslant \frac{1}{2}(3+n)$.

2.3. *Conditions of validity*

We now recall briefly the conditions of validity of the present theory. First we consider the conditions for neglecting the effect of viscosity (ν). The ratio between the inertial forces F_i and the viscous forces F_v can be written as $F_i/F_v \approx (t/t_{tr,n})^{(4\alpha-7-5n)/(3+n)}$ (for a detailed discussion see Huppert 1982); the characteristic time t_{tr} for which they become comparable is given by

$$t_{tr,n} = \left(\frac{\nu^{n+3}\,g^{2n+2}}{q^4}\right)^{\frac{1}{4\alpha-7-5n}}. \tag{19}$$

Then the critical value $\alpha_c = \frac{1}{4}(7+5n)$ separates the time intervals for which inertia dominates from those in which the viscosity is dominant. The inertial regime corresponds to

$$t \gg t_{tr} \quad (\alpha > \alpha_c); \qquad t \ll t_{tr} \quad (\alpha < \alpha_c). \tag{20}$$

In addition the thickness of the current must be always much smaller than its length to be consistent with shallow-water theory. But from (18), $h/x = bt^{\delta-2}\xi\Phi(\xi)$; then h/x will be bounded if $\delta < 2$, which requires

$$\alpha < 4+2n. \tag{21}$$

The same condition guarantees the consistency of the boundary condition at the front: the assumption of a quasi-static balance between the pressure difference and the drag due to the ambient fluid will be violated if the front acceleration increases without bounds. From $x_f(t) = bt^\delta\xi_f$ one obtains $\ddot{x}_f(t) = \delta(\delta-1)\,bt^{\delta-2}\xi_f$. Then as before we must have $\delta < 2$.

2.4. *Phase-plane formalism*

To derive the self-similar solutions we adapt the phase-plane formalism of gas dynamics (Sedov 1959; Courant & Friedrichs 1948) to shallow-water theory. We define

$$u = bt^{\delta-1}\delta\xi V(\xi), \quad h = b^2t^{2\delta-2}\delta^2\xi^2 Z(\xi). \tag{22}$$

Substituting in (12) one obtains after some algebra:

$$\frac{dZ}{dV} = \frac{\Delta_2(V,Z)}{\Delta_1(V,Z)}, \tag{23}$$

$$\frac{d(\ln\xi)}{dV} = \frac{\Delta(V,Z)}{\Delta_1(V,Z)}, \quad \frac{d(\ln\xi)}{dZ} = \frac{\Delta(V,Z)}{\Delta_2(V,Z)}, \tag{24}$$

† The non-dimensionalization of ξ and the form of δ have been chosen to satisfy the first source condition (14) (or, equivalently, the volume integral condition), as is standard in this type of problem.

with
$$\Delta(V,Z) = (1-V)^2 - Z, \tag{25}$$
$$\Delta_1(V,Z) = V(V-\mu)(1-V)+Z[(n+1)V+2(1-\mu)], \tag{26}$$
$$\Delta_2(V,Z) = Z\{(1-V)[2(V-\mu)+(n+1)V]+V(V-\mu)+2Z\}. \tag{27}$$

Notice that ξ is an autonomous variable in (23), (24). In terms of the phase variables (V,Z) the jump conditions are
$$V' = 1+2(V-1)/\phi(\mathscr{F}), \quad Z' = Z\phi(\mathscr{F})/2, \tag{28}$$
with $\mathscr{F} = (V-1)Z^{-\frac{1}{2}}$. From (28) one finds the relationship between the Froude numbers on each side of a jump:
$$\mathscr{F}' = \mathscr{F}[2/\phi(\mathscr{F})]^{\frac{3}{2}}. \tag{29}$$

The source boundary conditions are
$$\lim_{\xi\to 0}(2\pi\xi)^n\,\delta^3\xi^3 V(\xi)\,Z(\xi) = \alpha, \quad \lim_{\xi\to 0} V(\xi)\,Z(\xi)^{-\frac{1}{2}} = \mathscr{F}_0, \tag{30}$$
and the condition at the front is
$$V(\xi_f) = 1, \quad \beta^2 Z(\xi_f) = 1, \tag{31}$$
in which ξ_f denotes the self-similar coordinate of the front. The volume integral is
$$1 = (2\pi)^n\,\delta^2\int_0^{\xi_f}\xi^{2+n}Z(\xi)\,\mathrm{d}\xi. \tag{32}$$

The problem of finding the self-similar solutions is essentially reduced to solving the autonomous equation (23) for $Z(V)$; then a simple quadrature gives $\xi(V)$ from (24). Equation (23) is a single ODE linking the phase variables V, Z (related to the horizontal velocity and the depth of the current); its solutions are represented in the phase plane (V,Z) by curves (or portions thereof) called 'integral' or 'characteristic' curves. Only the half-plane $Z > 0$ is physically meaningful. A single integral curve passes through any regular point of the plane. The solution of a specific problem, characterized by its particular boundary (or initial) conditions, is represented by a piece (or pieces) of some integral curve (or curves). To find the appropriate curves it is necessary to know the behaviour of the solution in th neighbourhood of the singular points of (23).

In constructing the solutions it is important to notice that $\xi(V)$ has an extreme at the regular points of the parabola $Z = (1-V)^2$ (the 'critical parabola', $\mathscr{P}_{\mathscr{C}}$ for brevity) where $\Delta(V,Z) = 0$. If an integral curve crosses the $\mathscr{P}_{\mathscr{C}}$, $V(\xi)$ and $Z(\xi)$ are multivalued near the crossing, which cannot be allowed in a physical problem. Then the portion of a curve representing the solution of a real problem cannot cross the $\mathscr{P}_{\mathscr{C}}$ (except in the special case when the point of crossing is singular). The $\mathscr{P}_{\mathscr{C}}$ divides the phase plane in two regions: (*a*) the points (V,Z) below the $\mathscr{P}_{\mathscr{C}}$, which represent supercritical flows ($\mathscr{F} > 1$), and (*b*) those above the $\mathscr{P}_{\mathscr{C}}$, which represent subcritical flows ($\mathscr{F} < 1$).

As in gas dynamics, it is possible to obtain discontinuous solutions. This happens, for example, when the integral curve that represents the flow in a certain domain (i.e. in a certain ξ-interval) crosses the $\mathscr{P}_{\mathscr{C}}$ at a regular point. Then this integral curve must cease to represent the solution at some point J before the crossing. At this point the solution has a discontinuity, and its continuation beyond it must be represented by a portion of a different integral curve lying on the other side of the $\mathscr{P}_{\mathscr{C}}$, starting at another point J'; the phase variables on both sides of the discontinuity (i.e. the points J and

J') are related by the hydraulic jump conditions (28). A jump in a self-similar current occurs at a fixed value $\xi = \xi_J (= \xi_{J'})$ of the self-similar coordinate. Since a jump connects a supercritical flow with a subcritical one, the transition must go from points below the $\mathscr{P}_{\mathscr{C}}$ to points above it. Later we shall discuss how to find the conjugate points J and J' that define the jump.

3. Self-similar plane gravity currents

In what follows we shall consider in detail plane currents ($n = 0$, $\delta = 1/\mu = \frac{1}{3}(2+\alpha)$). It is convenient to consider separately the cases $0 < \alpha < 1$ ($\mu > 1$) and $\alpha > 1$ ($0 < \mu < 1$). (Actually only $\mu > \frac{1}{2}$ should be considered, since (21) requires $\alpha < 4$.) In addition there are two special cases of interest in which one obtains analytical solutions: $\alpha = 0$ ($\mu = \frac{3}{2}$), which corresponds to constant-volume currents, and $\alpha = 1$ ($\mu = 1$) which describes the currents produced by a source with constant outflow; they will be discussed separately in §5.

For $n = 0$, the autonomous equation (23) has eight singular points, of which three are at infinity (D, E, F). The position and properties of the singularities depend in general on μ (i.e. on α) and are summarized in table 1; the asymptotic behaviour of the solutions near a singularity determines their physical meaning. It is worth mentioning that for any given singularity S, the values $V = V_S$, $Z = Z_S$ represent an isolated exact self-similar solution of the shallow-water equations. Plots of the family of integral curves are shown in figure 1 for $\mu = \frac{2}{3}$ and in figure 2 for $\mu = \frac{4}{3}$, to illustrate the two different topologies of the phase plane for $\mu < 1$ and $\mu > 1$. The curves shown have been obtained by numerical integration of (23). The singular points of the autonomous equations at the finite are shown. All the curves that enter in the diagram from outside the (V, Z) range represented begin at the singular point F ($V_F = \infty$, $Z_F = \infty$). The singular point D is not relevant for the present problem, and the same is true for E, except for constant-volume currents ($\alpha = 0$, see §5).

According to the source boundary conditions (30), the self-similar currents must be represented by curves that begin at the node F (that represents a source at $x = 0$, see table 1). They must end at a (regular) point $P(V_P = 1, Z_P = 1/\beta^2)$ that represents the front; the position of the latter depends on the front boundary condition (31).

3.1. *Summary of and comment on GR's results*

It is convenient at this point to comment on the results of GR concerning plane gravity currents. Their treatment† is similar to the present one, but has two important differences. First GR do not include the second source condition (7) and secondly, except for a few general remarks, they do not consider solutions with jumps. As we see below these omissions have important consequences for the existence and uniqueness of solutions; we comment on these later. First however we summarize and comment on the arguments and conclusions of GR.

3.1.1. $\alpha > 1$ $(0 < \mu < 1)$

The integral curve passing through P does not arrive at F for certain values of β. It can be observed that there is a limiting integral curve $\mathscr{L}$ (see figure 1) which crosses the $\mathscr{P}_{\mathscr{C}}$ at F (i.e. for this curve $\mathscr{F}_0 = 1$). By numerical integration of (23) from F to $V = 1$

† In GR the phase variables V, Z are transformed into two variables V_1, W defined by $V_1 = 1/V$, $W = V^2/Z$. With this change F is transformed into the line $V_1 = 0$ of the (V_1, W)-plane. This change is not essential, and we prefer to discuss the problem using the (V, Z)-plane as is usually done in gas dynamics.

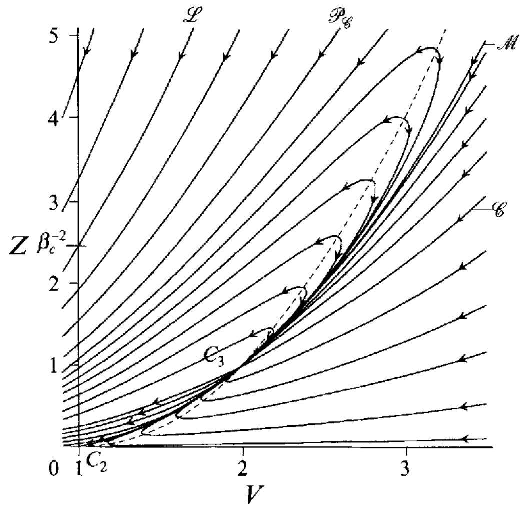

FIGURE 1. Integral curves for $\alpha = \frac{5}{2}$ ($\mu = \frac{2}{3}$), the arrows show the direction of increasing ξ.

				Behaviour		
Singularity	(V_S, Z_S)	μ	Type	$\xi =$	$u =$	$h =$
O	$(0,0)$	< 1	node	∞	0 or ∞	∞
		> 1	node	∞	0	0
B	$(2\mu/3, \mu^2/9)$	< 1	saddle	0	0	0
				∞	∞	∞
		> 1	node	∞	∞	∞
C_1	$(\mu, 0)$	< 1	logarithmic node	0	0	0
		> 1	logarithmic node	∞	∞	0
C_2	$(1,0)$	any	saddle	f	f	0
C_3	$(2,1)$	< 1	node	f	f	f
		> 1	saddle	f	f	f
D	$(\infty, 0)$	any	saddle	0	0	f
E	$(2(\mu-1), \infty)$	any	saddle	0	0	f
F	(∞, ∞)	any	node	0	f	f

TABLE 1. Singular points: position and properties of the singular points of the autonomous equation (23) for $n = 0$ ($\frac{1}{2} < \mu < \frac{3}{2}$). Here f represents a finite non-vanishing quantity

with the boundary condition $\mathscr{F}_0 = 1$ one can find the point P_c ($V_{P_c} = 1$, $Z_{P_c} = \beta_c^{-2}$, $\beta_c = \beta_c(\mu)$), and so the critical value β_c. For $\beta \leqslant \beta_c$ there will be integral curves joining F with P. In addition if $\beta > 2$ there will also be integral curves joining P with F, but such curves pass through the node C_3. But if $\beta_c < \beta < 2$ the curves starting at P cannot reach F. On this basis GR conclude that if $\beta_c < \beta < 2$ there are no solutions of (23) that satisfy the boundary conditions. They also conclude that for $\beta > 2$ the solution is not unique, since any curve through P arrives at C_3, and from there it can be continued to F in infinitely many ways.

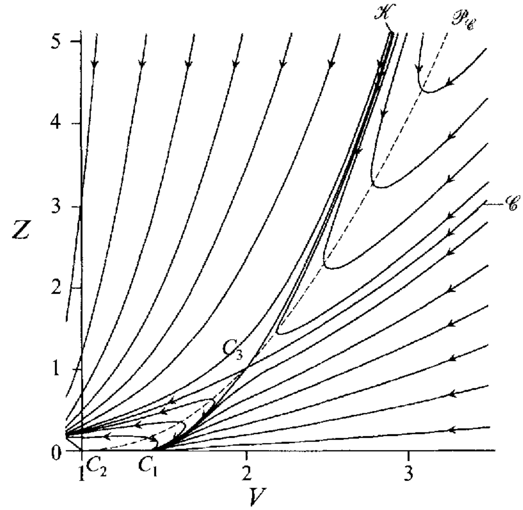

FIGURE 2. Integral curves for $\alpha = \frac{1}{4}$ ($\mu = \frac{4}{3}$), the arrows show the direction of increasing ξ.

3.1.2. $0 < \alpha < 1$ $(1 < \mu < \frac{3}{2})$

Now C_3 is a saddle. It can be seen from figure 2 that only when $\beta < 2$ will the curve through P arrive at F; these curves correspond to subcritical currents. If $\beta > 2$ the curves through P do not arrive at F. The conclusion of GR is that if $\beta > 2$ the problem has no solution.

3.1.3. *Comments*

The above arguments hold if we restrict ourselves to continuous solutions and the single source condition (6). However, it is hard to believe that no solutions exist for large β-intervals, and for other intervals the solution is not unique. First, it is contrary to intuition that self-similarity ceases to exist when the numerical value of a dimensionless parameter (β) changes, but remains finite and non-vanishing. On physical grounds, we expect that it must be possible to choose (within certain limits) the properties of the source (not only α, but also $\mathscr{F}_0$) regardless of the properties of the ambient fluid (i.e. β); notice that F is a node, which hints at this possibility. In particular, we fail to see why there should not be solutions with a supercritical flow near the source (which means that the source region of the current is not influenced by what happens at the front). Second, since the solution of any well-posed physical problem *must* be unique, the occurrence of many solutions (the infinite integral curves joining the nodes C_3 and F for $0 < \mu < 1$) clearly implies that some condition has been overlooked. As we show below the admission of discontinuous solutions resolves the questions of existence while the inclusion of the second source condition (7) deals with the uniqueness question since it fixes the curve through F that arrives at C_3. Finally we point out that the results of GR, that the only self-similar solutions for $\alpha = 1$ are uniform flows ($u = u_0 = (q_\alpha \beta^2)^{\frac{1}{3}}$, $h = h_0 = (q_\alpha/\beta)^{\frac{2}{3}}$), raises a further question since they do not yield any meaningful solution in the limit $\beta \rightarrow \infty$. Surely something is missing, because in this limit one should find (at least!) the well-known self-similar current describing the 'breaking' of a dam (analogous to the familiar gas dynamic self-similar expansion wave, see Zel'dovich & Raizer 1968, also Gratton 1991), see for example Whitham (1974).

3.2. *The different types of solutions and their construction*

Here and in the following Sections we shall show that for any β it is always possible to find solutions that are unique if the source is completely specified. In addition we shall use physical criteria to classify the solutions that appear for different ranges of β, $\mathscr{F}_0$. Ultimately a complete and satisfactory family of self-similar gravity currents is obtained, free from omissions and ambiguities.

Near the source the solution must be represented by an integral curve that begins at F, which we shall denote $\mathscr{C}_F$ ($\mathscr{C}_F$ depends on $\mathscr{F}_0$ as we shall now show). The integral trajectory must end at P (whose position depends on β), but may be discontinuous since the current may have hydraulic jumps. Let us first discuss how the source properties determine $\mathscr{C}_F$ and the complete solution in the source region. Integration of (23), (24) near F yields

$$Z = (V/\mathscr{F}_0)^2, \quad \xi = K/V, \quad K = \text{const.} \tag{33}$$

The constant K is a scale factor for ξ, and appears because the governing equations (23), (24) depend logarithmically on ξ. Using (33) in (30) one obtains

$$K^3 = \mu^2(3-2\mu)\,\mathscr{F}_0^2. \tag{34}$$

From (34) we see that α and $\mathscr{F}_0$ determine K so that the source boundary conditions determine uniquely the curve $\mathscr{C}_F$ and the complete solution near the source.

On the other hand, the front boundary condition (31) also determines uniquely an integral curve (the single curve passing through the regular point P) that we shall denote $\mathscr{C}_P$ ($\mathscr{C}_P = \mathscr{C}_P(\beta)$), which is always subcritical, and which may, or may not, coincide with $\mathscr{C}_F$. But notice that the complete solution near the front is not uniquely determined by β since ξ_f is not yet known. To determine ξ_f and so the complete solution near the front it is necessary first to find the adequate matching between the two parts of the solution, represented by $\mathscr{C}_F$ and $\mathscr{C}_P$.

Depending on the relation between $\mathscr{C}_F$ and $\mathscr{C}_P$, different situations can occur, which lead to different ways of constructing the solution and so to solutions of a different character.

3.2.1. *Continuous regular solutions (Type I)*

The simplest case occurs when $\mathscr{C}_F$ coincides with $\mathscr{C}_P$, and all its points are regular (except F). Then the solution is represented by a continuous integral curve FP. For each β there can be only one (or none) Type I solution, corresponding to a certain $\mathscr{F}_0$. Conversely, for each $\mathscr{F}_0$ there can be only one (or none) Type I solution, corresponding to a single β. It is characteristic of Type I solutions that the properties of the source ($\mathscr{F}_0$) and of the front (β) cannot be chosen independently, but must be compatible, so that $\mathscr{F}_0 = \mathscr{F}_0(\beta)$. It is important to notice that since $\mathscr{C}_P$ is always subcritical, a necessary (but not sufficient!) condition for the existence of Type I solutions is that $\mathscr{C}_F$ be subcritical (above the $\mathscr{P}_{\mathscr{C}}$), i.e. $\mathscr{F}_0 < 1$ (an exception to this occurs for $\alpha = 1$, see §5).

3.2.2. *Continuous solutions with a critical transition (Type II)*

A second possibility is that $\mathscr{C}_F$ and $\mathscr{C}_P$ join at a singular point. This occurs when both arrive at C_3. If this happens we shall have a continuous solution as in §3.2.1, but now for each β there can be an infinite number of solutions of this type since there will be a range of values of $\mathscr{F}_0$ for which $\mathscr{C}_F$ arrives at C_3 (before crossing the $\mathscr{P}_{\mathscr{C}}$). Each gives a continuous solution consisting of a piece FC_3 (that depends on $\mathscr{F}_0$) that is joined at

C_3 with a piece C_3P (that depends on β). For a given β, there can be an infinite number of solutions of this type, which represent currents whose fronts have equal properties, but are produced by sources having different $\mathscr{F}_0$. Likewise, for a given $\mathscr{F}_0$ such that $\mathscr{C}_F$ arrives at C_3, there can be an infinite number of Type II solutions, which represent currents produced by a given source, but whose fronts have different properties. Now, in contrast to above, $\mathscr{F}_0$ is not uniquely determined by β. The solutions that GR brand as 'not unique' belong to this class (actually GR found a family of solutions, corresponding to sources with different characteristics that produce currents with fronts having equal properties, or vice versa).

3.2.3. *Discontinuous solutions* (*Type III*)

It may be that $\mathscr{C}_F$ and $\mathscr{C}_P$ do not have points in common. When this happens, it is still possible to construct solutions, but they must be discontinuous. One possibility is to have a hydraulic jump connecting $\mathscr{C}_F$ with $\mathscr{C}_P$ (a different, less obvious possibility will be discussed below). Clearly, a necessary condition for the existence of solutions with jumps is that $\mathscr{C}_F$ be supercritical.

To discuss the construction of discontinuous solutions it is convenient to introduce a shorthand notation for the jump conjugates in the phase plane: let $X(V,Z)$ be a point, then $X' = X'(V',Z') = \mathscr{T}(X)$ is the point that is obtained applying the jump relations (28) to X. In the same way one can denote the conjugates of curves and regions of the phase plane, for example one has $\mathscr{P}'_{\mathscr{C}} = \mathscr{T}(\mathscr{P}_{\mathscr{C}}) = \mathscr{P}_{\mathscr{C}}$.

The prescription for the construction of a discontinuous solution is the following: let $\mathscr{C}_F$ represent the solution near the source, and let $\mathscr{C}_P$ represent the solution near the front. The intersection J of $\mathscr{C}_F$ and $\mathscr{T}(\mathscr{C}_P)$, if it exists, represents the variables before the jump. Then the intersection J' of $\mathscr{C}_P$ and $\mathscr{T}(\mathscr{C}_F)$, represents the variables after the jump (clearly, $J' = \mathscr{T}(J)$). The solution is then represented by the discontinuous pieces FJ and $J'P$. Notice that the position of J on the curve $\mathscr{C}_F$ depends on β; reciprocally, the position of J' on $\mathscr{C}_P$ depends on $\mathscr{F}_0$.

For each β there can be an infinite number of Type III currents, because there will be a range of $\mathscr{F}_0$ for which $\mathscr{C}_F$ can be connected by a jump with $\mathscr{C}_P$, thus allowing a solution to be constructed using the above prescription: $\mathscr{F}_0$ is not uniquely determined by β. As in the previous case we have families of currents produced by sources with different characteristics, but whose fronts have the same properties (or vice versa).

3.2.4. *Discontinuous solutions with a continuous critical transition* (*Type IV*)

When $\mathscr{C}_F$ and $\mathscr{C}_P$ do not have points in common and simultaneously $\mathscr{C}_F$ and $\mathscr{T}(\mathscr{C}_P)$ (or equivalently, $\mathscr{C}_P$ and $\mathscr{T}(\mathscr{C}_F)$) have no intersection, it may be still possible to construct a solution. In fact, we shall show below that for $\alpha < 1$ there is a special integral curve $\mathscr{K}$ that intersects both $\mathscr{T}(\mathscr{C}_F)$ and $\mathscr{T}(\mathscr{C}_P)$ at points J'_1, J_2, respectively. Then we can construct a solution represented by *three* discontinuous parts: the piece FJ_1 of $\mathscr{C}_F$, the piece J'_1J_2 of $\mathscr{K}$, and the piece J'_2P of $\mathscr{C}_P$. Since $\mathscr{K}$ is fixed, the position of J_1 on $\mathscr{C}_F$ will depend on $\mathscr{F}_0$, and the position of J'_2 on $\mathscr{C}_P$ will depend on β. Since $\mathscr{T}(\mathscr{C}_P)$ and $\mathscr{T}(\mathscr{C}_F)$ lie on different sides of the $\mathscr{P}_{\mathscr{C}}$, the curve $\mathscr{K}$ must cross the $\mathscr{P}_{\mathscr{C}}$ at a singular point to have solutions of this type. Then Type IV currents consist of a supercritical source part and a subcritical front part, linked by an intermediate region (represented by the piece J'_1J_2 of $\mathscr{K}$) in which the flow has a critical transition, passing from subcritical (near J'_1) to supercritical (near J_2). Hydraulic jumps connect the intermediate part of the current with its source and front parts.

As happens for Types II, III, for a given β there can be an infinite number of Type IV solutions, because there will be a range of $\mathscr{F}_0$ for which $\mathscr{C}_F$ can be connected by a jump with $\mathscr{K}$. These solutions will represent currents whose fronts have equal properties, but are produced by sources having different $\mathscr{F}_0$. Then $\mathscr{F}_0$ is not uniquely determined by β, and we shall have families of solutions that describe currents produced by sources with different characteristics, but whose fronts have the same properties (or vice versa).

Notice that the currents of all types (I–IV) are always subcritical in the front region. But in the source region they can be either subcritical or supercritical, depending on $\mathscr{F}_0$.

4. Existence of the solutions and their properties

We shall now find the ranges of $\alpha, \beta, \mathscr{F}_0$ in which the different types of currents appear and discuss their properties.

4.1. *Solutions for $\alpha > 1$ ($0 < \mu < 1$)*

The topology of the relevant part of the phase plane is shown in figure 1, and some of its important features are displayed in figure 3(*a*) to help the reader in following the discussion. The intervals of $\beta, \mathscr{F}_0$ in which the solutions are found can be visualized in figure 3(*b*). Notice that C_3 is a node, and that there are two integral curves having $\mathscr{F}_0 = 1$, namely the limiting curve $\mathscr{L}$ lying above the $\mathscr{P}_{\mathscr{C}}$ (which was already introduced in §3.1), and the curve $\mathscr{M}$ that lies below the $\mathscr{P}_{\mathscr{C}}$ and goes from F to C_3.

4.1.1. *Regular continuous solutions* (*Type I*)

If $0 < \beta \leqslant \beta_c$ the curve $\mathscr{C}_P$ does not cross the $\mathscr{P}_{\mathscr{C}}$ and arrives at F without passing through any singularity, coinciding with the $\mathscr{C}_F$ curve corresponding to a certain $\mathscr{F}_0 = \mathscr{F}_0(\beta) < 1$ ($\mathscr{F}_0(\beta)$ is determined by numerical integration of (23)). These are the continuous solutions discussed in GR, for which β determines uniquely $\mathscr{F}_0$. The physical interpretation is obvious: since the current is everywhere subcritical, the resistance of the ambient fluid (expressed by β) influences the entire flow up to the source; in consequence $\mathscr{F}_0$ cannot take any arbitrary value, but only that which is compatible with β.

4.1.2. *Continuous solutions with a critical transition* (*Type II*)

If $\beta \geqslant 2$ the curve $\mathscr{C}_P$ crosses the $\mathscr{P}_{\mathscr{C}}$ passing through C_3. Then the solution can be continued to F without the need of a hydraulic jump (it is allowed to cross the $\mathscr{P}_{\mathscr{C}}$ through a singularity) in an infinite number of ways, since all the $\mathscr{C}_F$ with $1 < \mathscr{F}_0 < 2$ arrive at C_3. These curves lie in the wedge-shaped region $\mathscr{R}$ of the phase plane limited by $\mathscr{M}, C_3$, and the analytical integral curve $\mathscr{C}$ (whose equation is $Z = V^2/4$). The current is supercritical near the source (piece FC_3) and subcritical in the piece C_3P. Here β does not determine uniquely $\mathscr{F}_0$. The physical interpretation is the following: since the current is supercritical near the source, the ambient fluid resistance (given by β) cannot influence the entire flow, but only its subcritical part (C_3P). The source region of the current (FC_3) is determined only by $\mathscr{F}_0$.

4.1.3. *Discontinuous solutions* (*Type III*)

This type of solution exists if the flow near the source is supercritical ($\mathscr{F}_0 > 1$). We distinguish two subcases:

(*a*) If $1 < \mathscr{F}_0 < 2$ the curve $\mathscr{C}_F$ lies in $\mathscr{R}$, and $\mathscr{C}'_F = \mathscr{T}(\mathscr{C}_F)$ lies in the wedge-shaped region $\mathscr{R}' = \mathscr{T}(\mathscr{R})$ above the $\mathscr{P}_{\mathscr{C}}$, limited by $\mathscr{C}' = \mathscr{T}(\mathscr{C})$, C_3 and $\mathscr{M}' = \mathscr{T}(\mathscr{M})$. Clearly, any curve $\mathscr{C}_P$ that enters $\mathscr{R}'$ can be connected by a jump to some $\mathscr{C}_F$, using the

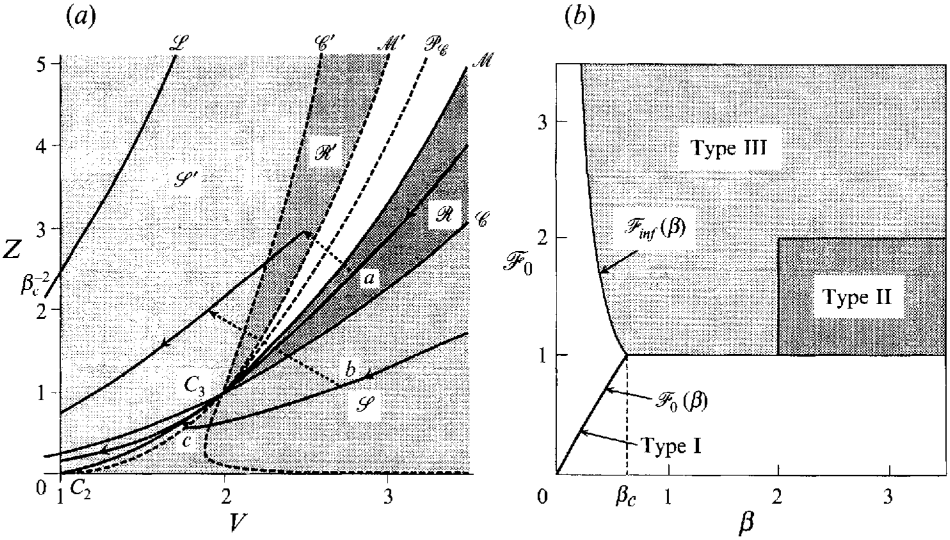

FIGURE 3. Case $\alpha > 1$: (*a*) relevant features of the phase plane and construction of solutions, the labels *a, b, c* denote Type III solutions; (*b*) intervals of parameters in which the different types of currents appear. The curves have been obtained by numerical integration for $\alpha = \frac{5}{2}$ ($\mu = \frac{2}{3}$).

prescription given in §3.2 (see the solution labelled *a* in figure 3*a*). But $\mathscr{C}_P$ will enter $\mathscr{R}'$ only if $\beta_{lim} < \beta < 2$, where β_{lim} is determined by the condition that $\mathscr{C}_P(\beta_{lim})$ crosses $\mathscr{C}'$ at F.† If $\beta < \beta_{lim}$ the curve $\mathscr{C}_P$ will never enter $\mathscr{R}'$; if $\beta > 2$ it crosses the $\mathscr{P}_{\mathscr{C}}$ through C_3 and does not enter $\mathscr{R}'$ (this case leads to a Type II current). It is easy to verify that $\beta_{lim} < \beta_c$. Furthermore, if $\beta_{lim} < \beta < \beta_c$ there will be Type III solutions only for $\mathscr{F}_0$ in the interval $\mathscr{F}_{inf}(\beta) < \mathscr{F}_0 < 2$, where $\mathscr{F}_{inf}(\beta)$ is the $\mathscr{F}_0$ value of the particular $\mathscr{C}_P$ that crosses at F the curve $\mathscr{T}(\mathscr{C}_P)$ corresponding to the β we are considering. Notice that $\mathscr{F}_{inf}(\beta) > 1$, since $\mathscr{F}_{inf}(\beta)$ and $\mathscr{F}(\beta)$ are related by (29). On the other hand, if $\beta_c < \beta < 2$, it is possible to find Type III solutions for $\mathscr{F}_0$ in the interval $1 < \mathscr{F}_0 < 2$.

(*b*) When $\mathscr{F}_0 > 2$, $\mathscr{C}_P$ lies in the region $\mathscr{S}$ of the phase plane limited from above by the $\mathscr{P}_{\mathscr{C}}$ and $\mathscr{C}$, and from below by $Z = 0$. All the $\mathscr{C}_P$ cross the $\mathscr{P}_{\mathscr{C}}$ before arriving at C_3. The region $\mathscr{S}' = \mathscr{T}(\mathscr{S})$ lies above $\mathscr{P}_{\mathscr{C}}$ and is limited by $V = 1$, the $\mathscr{P}_{\mathscr{C}}$ and $\mathscr{C}'$. It is easy to see that for any β, there are parts of $\mathscr{C}_P$ within $\mathscr{S}'$. Therefore it is possible to construct discontinuous solutions for any β (see the solutions labelled b, c in figure 3*a*). It can be shown that if $\beta > \beta_{lim}$ such solutions exist for any $\mathscr{F}_0 > 2$. On the other hand, if $\beta < \beta_{lim}$ there are Type III solutions only for $\mathscr{F}_0 > \mathscr{F}_{inf}(\beta)$.

In all the discontinuous solutions the source region FJ of the current is supercritical so that the effects of the ambient fluid resistance cannot propagate upstream to the source (β determines the current profile only in its subcritical part, while the rest is determined only by $\mathscr{F}_0$).

The intervals of β and $\mathscr{F}_0$ corresponding to the different types of currents are shown in figure 3(*b*). Notice that there are solutions corresponding to $\beta = \infty$ (i.e. currents in a vacuum); but then the front part of the current is represented by the special integral curve joining C_2 with C_3. It is interesting to observe that if $0 < \beta \leqslant \beta_c$ it is possible to have for the same β (but different $\mathscr{F}_0$ values) a continuous solution (with $\mathscr{F}_0 < 1$) and infinite discontinuous solutions (with $\mathscr{F}_0 > \mathscr{F}_{inf}(\beta)$). These currents are all identical in the front region, but different in the source region. In addition the discontinuous

† It can be verified that $\mathscr{C}_P(\beta_{lim})$ corresponds to $\mathscr{F}_0 = 2^{\frac{5}{3}}[\sqrt{33}-1]^{-\frac{3}{2}}$.

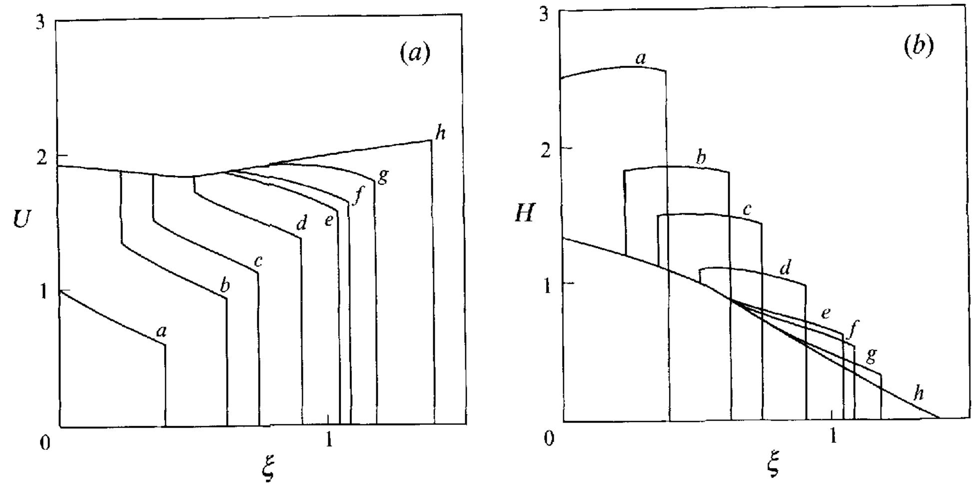

FIGURE 4. Currents for $\alpha = \frac{5}{2}$ $(\mu = \frac{2}{3})$, $1 < \mathscr{F}_0 < 2$ and β variable: (*a*) dimensionless velocity profiles $U(\xi)$, (*b*) dimensionless depth profiles $H(\xi)$. The solutions correspond to $\mathscr{F}_0 = 1.66$, $\beta_a = 0.26$, $\beta_b = 0.88$, $\beta_c = 1.10$, $\beta_d = 1.75$, $\beta_e = 2$, $\beta_f = 2.24$, $\beta_g = 3.16$, $\beta_h = \infty$. The solution marked *a* has a hydraulic jump at $x = 0$, and is, in effect, a Type I current.

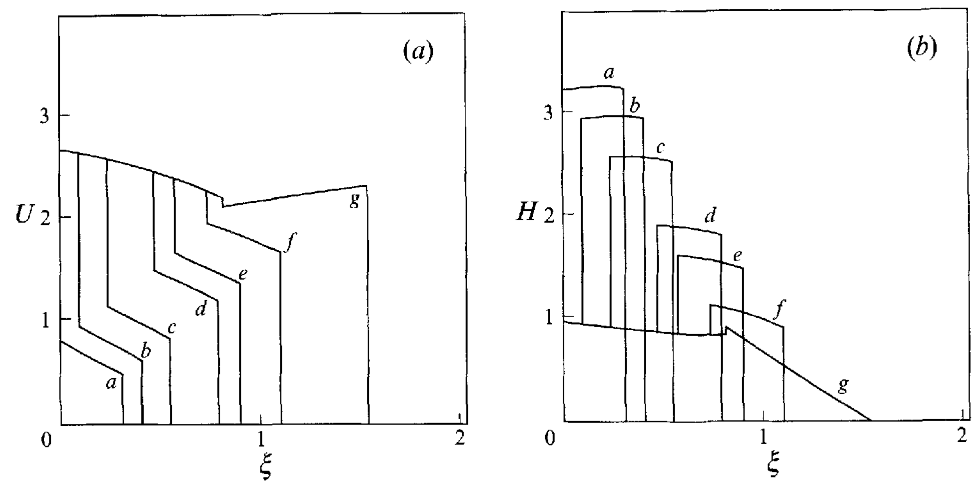

FIGURE 5. Currents for $\alpha = \frac{5}{2}$ $(\mu = \frac{2}{3})$, $\mathscr{F}_0 > 2$ and β variable: (*a*) dimensionless velocity profiles $U(\xi)$, (*b*) dimensionless depth profiles $H(\xi)$. The solutions correspond to $\mathscr{F}_0 = 2.76$, $\beta_a = 0.26$, $\beta_b = 0.35$, $\beta_c = 0.51$, $\beta_d = 0.88$, $\beta_e = 1.10$, $\beta_f = 1.75$, $\beta_g = \infty$. The solution marked *a* has a hydraulic jump at $x = 0$.

solutions differ among themselves regarding the position of the jump, which is very near the source for $\mathscr{F}_0$ close to $\mathscr{F}_{inf}(\beta)$, and moves towards the front as $\mathscr{F}_0$ increases. When $\mathscr{F}_0 = \mathscr{F}_{inf}(\beta)$ the jump occurs precisely at the source, and this 'discontinuous' solution coincides with the continuous (Type I) solution corresponding to the β in question, whose $\mathscr{F}_0 = \mathscr{F}(\beta)$ is conjugate to $\mathscr{F}_{inf}(\beta)$ by (29). In other words, the continuous solutions can be considered as the limit of discontinuous solutions as the jump approaches and chokes the source.

Some solutions obtained by numerical integration of (23), (24) for $\alpha = \frac{5}{2}$ are shown in figures 4 and 5, which show the dimensionless depth profiles $H(\xi) = \delta^2\xi^2 Z(\xi) = h/b^2t^{2\delta-2}$ and the dimensionless velocity profile $U(\xi) = \delta\xi V(\xi) = u/bt^{\delta-1}$. Figure 4(*a*, *b*) represents currents produced by a moderately supercritical source ($\mathscr{F}_0 = 1.66$) and different β, which corresponds to a cut of the diagram of figure 3(*b*) along an horizontal line. Then as β increases one finds first a choked current (Type I), then Type III currents for $\beta < 2$, and finally Type II flows. Figure 5(*a*, *b*) represents currents produced by a strongly supercritical source ($\mathscr{F}_0 = 2.76$) and different β. As β increases (see figure 6) one finds first a choked current (Typc I), then Type III currents, but no Type II flows appear. In both cases it can be observed that the length of the current increases and its thickness at the front decreases as β is increased (i.e. the ambient fluid resistance is diminished), according to intuition. Except for the jumps, the thickness of the current decreases as ξ increases: this is because the parts of the current farther from the source are made up from the fluid that issued earlier, when the source flow was smaller ($\alpha > 1$). It can also be observed that the velocity is nearly constant in the supercritical region, but (except for $\beta \to \infty$) decreases significantly with increasing ξ in the subcritical part near the front; also the length of this part is almost independent of β.

4.2. *Solutions for* $\alpha < 1$ $(1 < \mu < \frac{3}{2})$

Now C_3 is a saddle, and there are only two integral curves that cross the $\mathscr{P}_{\mathscr{C}}$ passing through C_3. One is $\mathscr{C}$ (see §4.1.2), which represents a special continuous solution that exists only for $\beta = 2$ (see §5). The other, which we call $\mathscr{K}(\mu)$, goes from C_3 to F passing above the $\mathscr{P}_{\mathscr{C}}$ and corresponds to $\mathscr{F}_0 = \mathscr{F}_{\mathscr{K}}$ (the unique curve $\mathscr{K}(\mu)$ can be determined by numerical integration of (23) starting from C_3, where its slope is given by $(5-\mu)/2(2-\mu)$; in this way the value of $\mathscr{F}_{\mathscr{K}} = \mathscr{F}_{\mathscr{K}}(\mu) < 1$ is also obtained). Its conjugate $\mathscr{K}' = \mathscr{T}(\mathscr{K})$ allows another limiting value of $\mathscr{F}_0$ to be defined, which we shall denote $\mathscr{F}_{\mathscr{K}'}$ ($\mathscr{F}_{\mathscr{K}'} = \mathscr{F}_{\mathscr{K}'}(\mu), 1 < \mathscr{F}_{\mathscr{K}'} < 2$). (From (29) one can obtain $\mathscr{F}_{\mathscr{K}'} = \mathscr{F}_{\mathscr{K}}[2/\phi(\mathscr{F}_{\mathscr{K}})]^{\frac{3}{2}}$.) The relevant part of the phase plane is displayed in figure 2, and some of its important features are shown in figure 6(*a*) to help the reader. In this figure three $\mathscr{C}_P$ curves ($\mathscr{C}_P, \mathscr{C}'_P, \mathscr{C}''_P$) have been drawn, corresponding to different β-values. The intervals of $\beta, \mathscr{F}_0$ in which the solutions are found are shown in figure 6(*b*).

4.2.1. *Continuous regular solutions* (*Type I*)

When $\beta < 2$ the curve $\mathscr{C}_P$ arrives at F without crossing the $\mathscr{P}_{\mathscr{C}}$. The same considerations as presented in §4.1.1 apply in this case. These solutions were found by GR, and for them $\mathscr{F}_0 = \mathscr{F}_0(\beta) < \mathscr{F}_{\mathscr{K}} < 1$.

4.2.2. *Discontinuous solutions* (*Type III*)

Depending on $\mathscr{F}_0$ there are two cases:

(*a*) If $\mathscr{F}_{\mathscr{K}'} < \mathscr{F}_0 < 2$ the curve $\mathscr{C}_F$ lies in the region $\mathscr{R}$, limited by the $\mathscr{P}_{\mathscr{C}}$, C_3 and $\mathscr{C}$. But in contrast to the case $\alpha > 1$, $\mathscr{C}_F$ does not arrive at C_3, but crosses the $\mathscr{P}_{\mathscr{C}}$ above this point. Its conjugate ($\mathscr{C}_{F'} = \mathscr{T}(\mathscr{C}_F)$) lies in $\mathscr{R}'$. Only those $\mathscr{C}_P$ that enter in the subregion $\mathscr{R}_{\mathscr{K}'}$ of $\mathscr{R}'$ limited by $\mathscr{K}$ and $\mathscr{C}'$ can connect by a hydraulic jump with those $\mathscr{C}_F$ that have parts in the subregion $\mathscr{R}_{\mathscr{K}}$ of $\mathscr{R}$ limited from above by $\mathscr{K}'$ and from below by $\mathscr{C}$ (see the solution labelled *a* in figure 6*a*). Now, $\mathscr{C}_P$ enters into $\mathscr{R}_{\mathscr{K}'}$ only if $\beta_{lim} < \beta < 2$; in this case it can connect with the $\mathscr{C}_F$ corresponding $\mathscr{F}_0 > \mathscr{F}_{inf}(\beta) \geqslant \mathscr{F}_{\mathscr{K}'}$; if $\beta < \beta_{lim}$ the curve $\mathscr{C}_P$ never enters $\mathscr{R}_{\mathscr{K}'}$; if $\beta > 2$ it crosses the $\mathscr{P}_{\mathscr{C}}$ below C_3 and does not enter $\mathscr{R}_{\mathscr{K}'}$, either.

(*b*) When $\mathscr{F}_0 > 2$ the curves $\mathscr{C}_F$ lie in $\mathscr{S}_{\mathscr{K}}$ (limited by $\mathscr{C}$, C_3, $\mathscr{K}$, C_1 and $Z = 0$) and join F with C_1. For any β, parts of $\mathscr{C}_P$ are within $\mathscr{S}_{\mathscr{K}'}$ so that it is always possible to find discontinuous solutions following the usual prescription provided $\mathscr{F}_0 > [\mathscr{F}_{inf}(\beta), 2]$ (see the solution labelled *b* in figure 6*a*).

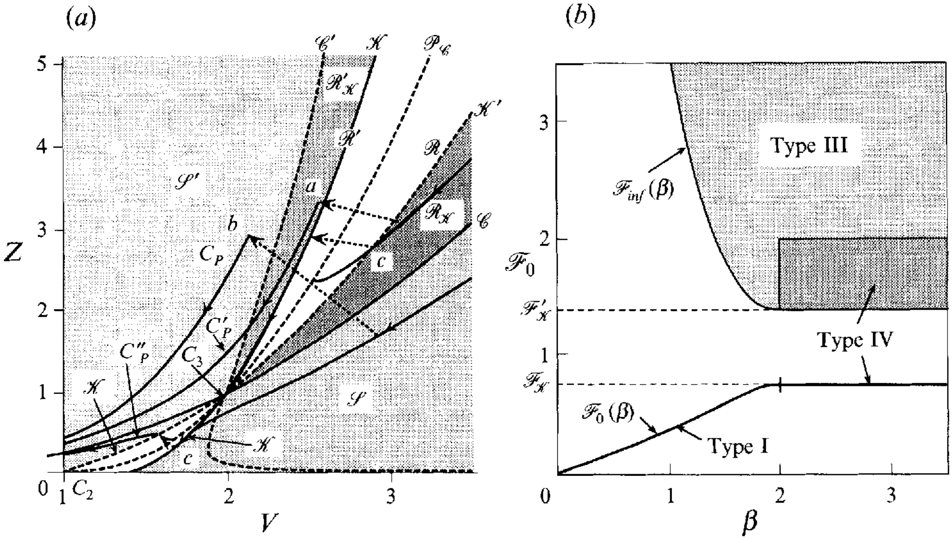

FIGURE 6. Case $\alpha < 1$: (a) relevant features of the phase plane and construction of solutions, the labels a, b denotes Type III solutions, c is a Type IV solution; (b) intervals of parameters in which the different types of currents appear. The curves have been obtained by numerical integration for $\alpha = \frac{1}{4}$ ($\mu = \frac{4}{3}$).

Concerning the properties of Type III solutions, the discussion in §4.1.3 also applies. When $\mathscr{F}_0 = \mathscr{F}_{inf}(\beta)$ the jump occurs precisely at the source, and this 'discontinuous' solution coincides with the continuous (Type I) solution corresponding to the β in question, whose $\mathscr{F}_0 = \mathscr{F}(\beta) = \mathscr{F}_{inf}[2/\phi(\mathscr{F}_{inf})]^{\frac{3}{2}}$ is conjugate to $\mathscr{F}_{inf}(\beta)$ by (29).

4.2.3. *Discontinuous solutions with a continuous critical transition* (*Type IV*)

When $\mathscr{F}_{\mathscr{K}'} < \mathscr{F}_0 < 2$, $\mathscr{T}(\mathscr{C}_P)$ intersects the special integral curve $\mathscr{K}$, whose conjugate $\mathscr{K}'$ (after crossing the $\mathscr{P}_{\mathscr{C}}$ at C_3) in turn intersects all the curves $\mathscr{C}_P$ with $\beta > 2$. Then Type IV solutions can be constructed following the prescription given in §3.2 (see the solution labelled c in figure 6a).

A special limiting case of Type IV solutions is obtained when the first jump occurs precisely at the source; the solutions are represented by the piece FJ_2 of $\mathscr{K}$ and $J_2'P$ of $\mathscr{C}_P$ for any $\beta > 2$ and all correspond to $\mathscr{F}_0 = \mathscr{F}_{\mathscr{K}}$.

The intervals of β and $\mathscr{F}_0$ corresponding to the different types of solutions are shown in figure 6(b). Like in the $\alpha > 1$ case, we find that if $0 < \beta \leqslant 2$ it is possible to have for the same β (but different $\mathscr{F}_0$ values) a continuous solution (with $\mathscr{F}_0 < 1$) and infinite discontinuous solutions (with $\mathscr{F}_0 > 2$ for $0 < \beta \leqslant \beta_{lim}$, and with $\mathscr{F}_0 > \mathscr{F}_{\mathscr{K}'}$ for $\beta_{lim} < \beta < 2$), all identical near the front, but different in the source region; the discontinuous solutions differ regarding the position of the jump, which moves towards the front as $\mathscr{F}_0$ increases. In contrast to the $\alpha > 1$ case, when $\alpha < 1$ there are no solutions for $\beta = \infty$ (currents in a vacuum). Also notice that while for any (finite) β it is always possible to find some solutions, the analogous situation is not true for $\mathscr{F}_0$, since no solutions exist in the interval $\mathscr{F}_{\mathscr{K}} < \mathscr{F}_0 < \mathscr{F}_{\mathscr{K}'}$.

It can be noticed that for $\alpha < 1$ there are no Type II currents since C_3 is now a saddle. Their place is taken by Type IV flows that occur also for $\beta < 2$, $\mathscr{F}_0 < 2$. The change of nature of C_3 as one passes from $\alpha > 1$ to $\alpha < 1$ explains this fact.

Various solutions for $\alpha = \frac{1}{4}$, obtained by numeric integration, are shown in figures 7 and 8, in which we represent the dimensionless depth profile $H(\xi) = \delta^2\xi^2 Z(\xi) =$

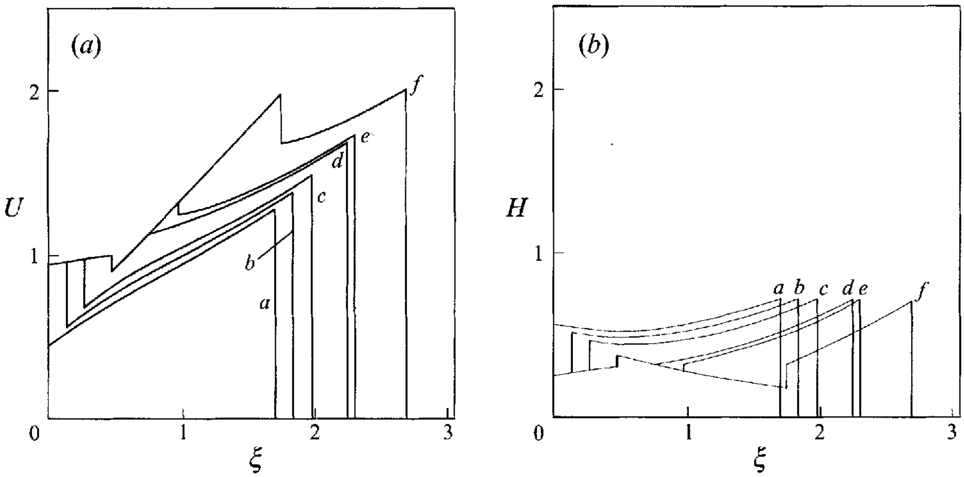

FIGURE 7. Currents for $\alpha = \frac{1}{4}$ ($\mu = \frac{4}{3}$), $1 < \mathscr{F}_0 < 2$ and β variable: (*a*) dimensionless velocity profiles $U(\xi)$, (*b*) dimensionless depth profiles $H(\xi)$. The solutions correspond to $\mathscr{F}_0 = 1.85$, $\beta_a = 1.49$, $\beta_b = 1.62$, $\beta_c = 1.74$, $\beta_d = 2$, $\beta_e = 2.05$, $\beta_f = 2.39$. The solution labelled *a* has a hydraulic jump at $x = 0$.

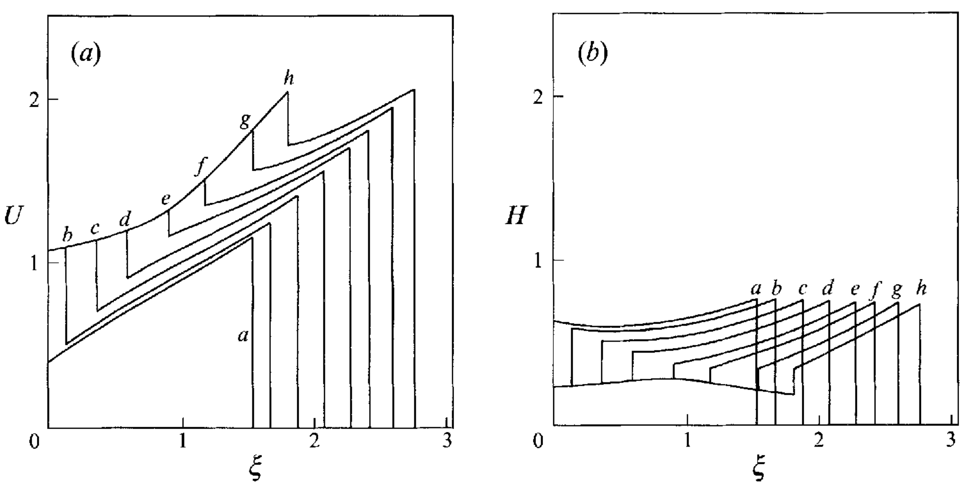

FIGURE 8. Currents for $\alpha = \frac{1}{4}$ ($\mu = \frac{4}{3}$), $\mathscr{F}_0 > 2$ and β variable: (*a*) dimensionless velocity profiles $U(\xi)$, (*b*) dimensionless depth profiles $H(\xi)$. The solutions correspond to $\mathscr{F}_0 = 2.26$, $\beta_a = 1.32$, $\beta_b = 1.43$, $\beta_c = 1.62$, $\beta_d = 1.80$, $\beta_e = 1.98$, $\beta_f = 2.10$, $\beta_g = 2.27$, $\beta_g = 2.27$, $\beta_h = 2.41$. The solution labelled *a* has a hydraulic jump at $x = 0$.

$h/b^2t^{2\delta-2}$ and the dimensionless velocity profile $U(\xi) = \delta\xi V(\xi) = u/bt^{\delta-1}$. Figure 7(*a*, *b*) represents currents produced by a moderately supercritical source ($\mathscr{F}_0 = 1.85$) and different β, which corresponds to a cut of the diagram of figure 6(*b*) along an horizontal line. As β increases one finds a choked current (Type I), then Type III currents for $\beta < 2$, and finally Type IV flows. Figure 8(*a*, *b*) represents currents produced by a strongly supercritical source ($\mathscr{F}_0 = 2.26$) and different β. As β increases (see figure 3) one finds first a choked current (Type I), then Type III currents, but no Type IV flows appear. In both cases it can be observed that the length of the current increases as β is increased (i.e. the ambient fluid resistance is diminished), according to intuition. Notice, however, that now (contrary to the $\alpha > 1$ case) the thickness at the front is nearly independent of β. In the supercritical part, as before, the thickness of the current varies very little with ξ; but in the subcritical part it increases rapidly with ξ. Most of the fluid tends to accumulate near the front: this is because the parts of the current farther from the source are made up from the fluid that issued earlier, when the source flow was larger ($\alpha < 1$). It can also be observed that the velocity increases very rapidly with ξ in the subcritical part near the front, contrary to what happens for $\alpha > 1$. The length of this part diminishes with β.

5. Special analytical solutions

5.1. *The spreading of a constant volume of fluid* ($\alpha = 0, \mu = \frac{3}{2}$)

The currents corresponding to $\alpha = 0, \beta < 2$, were previously known (see GR). In addition there are solutions for $\beta \geqslant 2$ as we shall presently show.

When $\alpha = 0$ the integral curve of interest is $V = 1$, on which lie three singular points (see table 1): $E(V_E = 1, Z_E = \infty)$, $B(V_B = 1, Z_B = \frac{1}{4})$ and $C_1(V_{C1} = \frac{3}{2}, Z_{C1} = 0)$. The singularity that represents $\xi = 0$ (that is *not* a source now) is E. The point B represents $\xi = \infty$, and C_1 represents a moving point of the current where the depth vanishes, and where $\xi = \xi_0 = \text{const}$. From (24) and $V = 1$ one obtains

$$\xi = K|Z - \tfrac{1}{4}|^{-\frac{1}{2}}, \quad K = \text{const}. \tag{35}$$

In consequence the solutions represented by pieces of $V = 1$ above B are of the form

$$V = 1, \quad Z = \tfrac{1}{4} + (K/\xi)^2, \quad (Z > \tfrac{1}{4}), \tag{36}$$

and those represented by pieces below B are given by

$$V = 1, \quad Z = \tfrac{1}{4} - (K/\xi)^2 \quad (Z < \tfrac{1}{4}). \tag{37}$$

Then the following cases can occur:

(*a*) When $\beta < 2$, P is above B and the solution is represented by the segment EP; setting

$$K = \tfrac{1}{2}\xi_f(4/\beta^2 - 1)^{\frac{1}{2}}, \tag{38}$$

one obtains
$$u = 2x/3t, \quad h = (x/3t)^2[1 + (4/\beta^2 - 1)(\xi_f/\xi)^2], \tag{39}$$

with
$$\xi_f = \left(\frac{27\beta^2}{12 - 2\beta^2}\right)^{\frac{1}{3}}. \tag{40}$$

These are the solutions of Fannelop & Waldman (1972) and Hoult (1972), which GR rederived with the phase-plane formalism. The depth of the current at $x = 0$ is

$$h(x = 0) = (b\xi_f/3)^2 (4/\beta^2 - 1)\, t^{-\frac{2}{3}}; \tag{41}$$

$h(x = 0)$ is less than the depth at the front, given by

$$h(x = x_f) = (b\xi_f/3)^2 (4/\beta^2)\, t^{-\frac{2}{3}}. \tag{42}$$

As β increases and approaches 2 the length of the current increases and $h(x = 0)$ diminishes.

(*b*) For $\beta = 2$ the current is described by the special isolated solution corresponding to the singular point B. One has $V = 1$, $Z = \frac{1}{4}$, and from (32) one finds $\xi_f = 3$. The solution is

$$u = 2x/3t, \quad h = (x/3t)^2, \quad x < x_f. \tag{43}$$

Notice that this current has vanishing depth at $x = 0$.

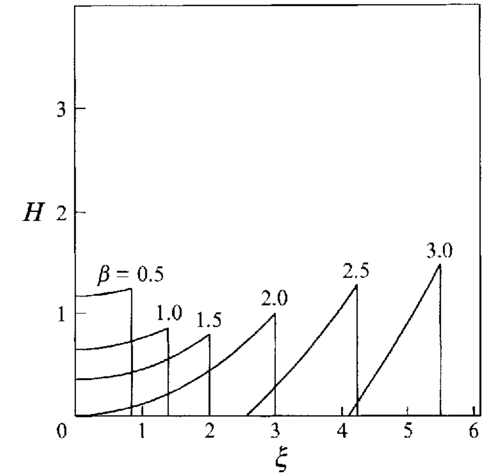

FIGURE 9. Dimensionless depth profiles $H(\xi)$ of the currents produced by the spreading of a constant volume of fluid ($\alpha = 0$). The curves are labelled according to the value of β.

(*c*) When $\beta > 2$, P lies below B. In consequence the solution cannot be extended to $\xi = 0$ (i.e. to E). Yet we can find a solution by considering the piece $C_1 P$ of the line $V = 1$. Then, using (37) and setting

$$K = \tfrac{1}{2}\xi_0, \quad \xi_0 = \xi_f(1-4/\beta^2)^{\frac{1}{2}}, \tag{44}$$

we find

$$u = 2x/3t, \quad h = (x/3t)^2[1-(\xi_0/\xi)^2], \quad \xi_0 \leqslant \xi \leqslant \xi_f \tag{45}$$

with

$$\xi_f = \left(\frac{27\beta^2}{12-2\beta^2[1-(1-4/\beta^2)^{\frac{3}{2}}]}\right)^{\frac{1}{3}}. \tag{46}$$

The depth of the current is zero in the interval $0 < \xi < \xi_0$.

Depth profiles of the currents for $\alpha = 0$ are shown on figure 9 for several values of β. It can be observed that as β increases the fluid tends to accumulate near the front and at the same time the length of the current increases, according to intuition. For $\beta > 2$ the dry region near the origin can be clearly observed.

To conclude this discussion it remains to mention that there is no self-similar solution of this kind in the limit $\beta \rightarrow \infty$ (spreading in a vacuum). The spreading of a constant volume of a liquid in a vacuum can be studied by the method of characteristics, and has no self-similar asymptotics (Gratton 1988); the result is completely analogous to the free expansion of a finite mass of a gas (see for example Stanyukovich 1960).

5.2. *Currents produced by a constant source* ($\alpha = 1, \mu = 1$)

The solution of the equivalent problem in gas dynamics is described in detail in Gratton (1991). If we set $\mu = 1$ in (23), a common factor Δ in the numerator and denominator cancels, and we obtain $\mathrm{d}Z/\mathrm{d}V = 2Z/V$ which upon integration yields

$$Z = (V/\mathscr{F}_0)^2. \tag{47}$$

Then integrating (24) and using (34) one finds

$$\xi = \mathscr{F}_0^{\frac{2}{3}}/V. \tag{48}$$

The integral curves are then a family of parabolas whose vertex is at the origin of the phase plane. It is easy to see that (47), (48) represent uniform flows with constant velocity $u = u_0 = \mathscr{F}_0^{\frac{2}{3}}$ and constant depth $h = h_0 = \mathscr{F}_0^{-\frac{2}{3}}$. Notice that for $\mu = 1$ one has $\Delta_1 = -V\Delta$, $\Delta_2 = -2Z\Delta$ so that $\xi(V)$, $\xi(Z)$ are not extreme on the $\mathscr{P}_{\mathscr{C}}$. Then it is allowed to cross the $\mathscr{P}_{\mathscr{C}}$ following the curves (47).

In addition it can be verified that for $\mu = 1$ the $\mathscr{P}_{\mathscr{C}}$ is a special integral curve, along which one has

$$Z = (V-1)^2, \quad \xi = K|V-\tfrac{2}{3}|^{-1}, \quad K = \text{const.} \tag{49}$$

Equation (49) represents a critical transition ($\mathscr{F} = 1$) between regions of different depth and velocity; it is analogous to an expansion or compression wave in a gas. Notice that in (49) ξ increases as one moves on the $\mathscr{P}_{\mathscr{C}}$ from F to B ($V_B = \frac{2}{3}, Z_B = \frac{1}{9}$) where $\xi = \infty$.

Then in the case $n = 0$, $\alpha = 1$, the self-similar solutions will comprise the uniform flows already discussed by GR, plus a variety of other not so trivial currents consisting of regions of uniform flow connected by hydraulic jumps and critical transitions. To find the currents we want it will suffice to impose the boundary conditions (30), (31) and when necessary, the jump conditions (28) following the prescriptions of §3.

5.2.1. *Uniform currents*

Using (30), (31), (47), (48) one finds $\mathscr{F}_0 = \beta$, $\xi_f = \beta^{\frac{2}{3}}$, so that the solution is given by

$$u = u_0 = (q_1\beta^2)^{\frac{1}{3}}, \quad h = h_0 = (q_1/\beta)^{\frac{2}{3}}, \quad x_f(t) = u_0 t. \tag{50}$$

Notice that there are continuous solutions for any β, including currents supercritical near the source (it is allowed to cross the $\mathscr{P}_{\mathscr{C}}$). Equations (50) are Type I currents, like those discussed in §4. Now they also exist for $\mathscr{F}_0 > 1$, but then they have a critical transition, which means that the source is outside the region of influence of the front.

5.2.2. *Solutions with critical transitions*

Consider a uniform current near the source represented by a curve $\mathscr{C}_F$ of the form (47) corresponding to $\mathscr{F}_0 > 1$; this curve will cross the $\mathscr{P}_{\mathscr{C}}$ at a point T. Clearly it is possible to find a family of solutions formed by: (i) the piece FT of $\mathscr{C}_F$, (ii) a piece TT' of the $\mathscr{P}_{\mathscr{C}}$ (where T' lies between T and B, since ξ must increase), and (iii) a piece $T'P$ of a uniform current corresponding to $\mathscr{F}'_0 > \mathscr{F}_0$. Then if TP represents a uniform current for a given β, it is possible to find an infinite number of solutions with critical transitions that have the piece FT in common with the uniform current, but that near the front have a different behaviour, corresponding to another uniform current with $\beta' > \beta$. The matching between both uniform parts occurs by a critical transition, represented by a piece of the $\mathscr{P}_{\mathscr{C}}$. A particular case of this kind corresponds to $\beta = \infty$, and consists of the part of the $\mathscr{P}_{\mathscr{C}}$ that joins F with the point $(1, 0)$; it is the solution of the classical problem of the breaking of a dam (for details see, for example, Whitham 1974). These currents are the analogues of the Type II currents discussed in §4.

5.2.3. *Discontinuous solutions*

Starting from a uniform current in the source region, represented by a curve $\mathscr{C}_F$ corresponding to $\mathscr{F}_0 > 1$, it is also possible to find a family of discontinuous solutions represented by: (i) a piece FJ of $\mathscr{C}_F$ that goes from F to a point J (J must lie below the $\mathscr{P}_{\mathscr{C}}$) where there is a hydraulic jump, and (ii) the piece $J'P$ (with $J' = \mathscr{T}(J)$) of another uniform current corresponding to $\mathscr{F}'_0 < \mathscr{F}_0$. Then if TP represents a uniform current for a given β, there is an infinite number of discontinuous solutions that have part of the piece FT in common with it, but that near the front have a different behaviour,

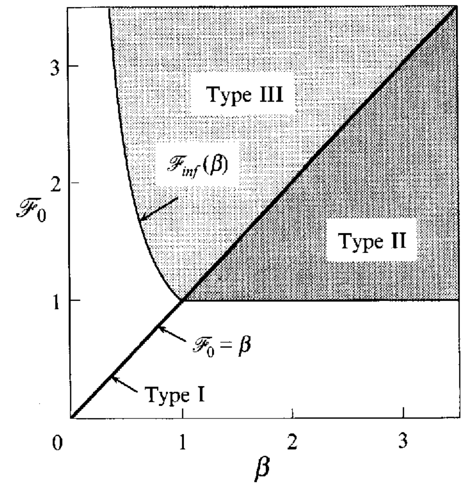

FIGURE 10. Currents produced by a source with constant inflow ($\alpha = 1$): intervals of parameters in which the different type of currents appear.

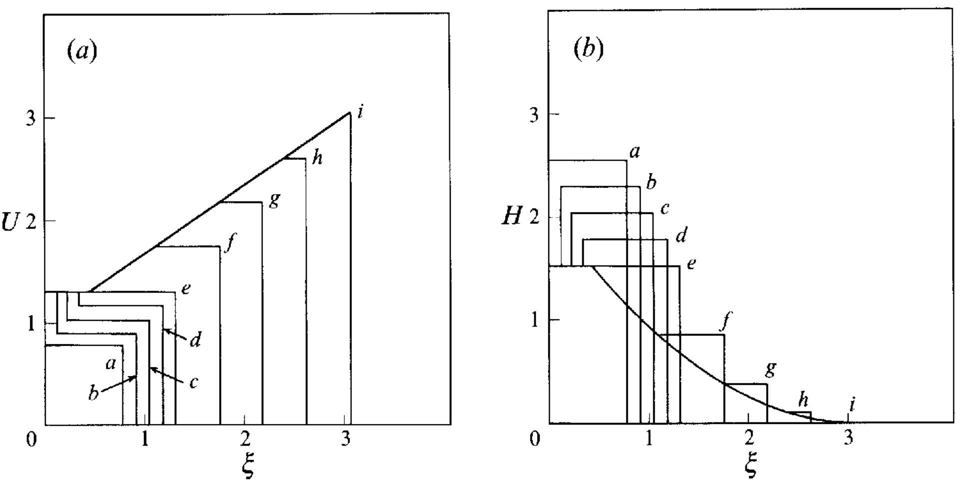

FIGURE 11. Currents produced by a source with constant inflow ($\alpha = 1$): (a) dimensionless velocity profiles $U(\xi)$, (b) dimensionless depth profiles $H(\xi)$. The solutions correspond to $\mathscr{F}_0 = \frac{3}{2}$, $\beta_a = 0.69$, $\beta_b = 0.84$, $\beta_c = 1.03$, $\beta_d = 1.24$, $\beta_e = 1.5$, $\beta_f = 2.67$, $\beta_g = 5$, $\beta_h = 12$, $\beta_i = \infty$. The solution labelled a has a hydraulic jump at $x = 0$.

corresponding to uniform currents with any other $\beta' < \beta$. The uniform parts are connected by a hydraulic jump. For $\beta > 1$ there are solutions of this kind for any $\mathscr{F}_0 > \beta$. For $\beta < 1$, there are solutions with jumps for $\mathscr{F}_0 > \mathscr{F}_{inf} = \beta[2/\phi(\beta)]^{\frac{1}{2}}$.

In figure 10 we represent the intervals of β and $\mathscr{F}_0$ corresponding to the different types of solutions. In figure 11 the profiles of U and H of some solutions for $\mathscr{F}_0 = 1.85$ and different β are displayed. They correspond to a cut of the diagram of figure 10 along a horizontal line. As β increases one find a choked current (Type I), then Type III, and finally Type II flows. As always, the length of the current increases as β is increased. The thickness at the front decreases with β. The velocity increases very rapidly with ξ in the critical transition region, whose length increases rapidly with β. It is interesting to compare the profiles of figure 11 with those corresponding to currents with $\alpha > 1$ (figures 4, 5) and $\alpha < 1$ (figure 7, 8) and notice the similarities and

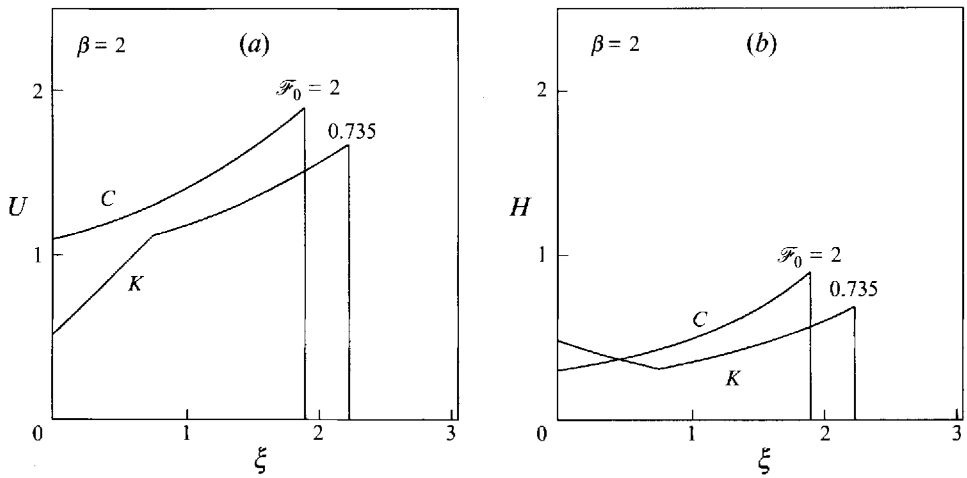

FIGURE 12. Solutions associated with the integral curve $\mathscr{C}$: (*a*) dimensionless velocity profiles $U(\xi)$, (*b*) dimensionless depth profiles $H(\xi)$. The labels C, K denote the special analytical solution with $\mathscr{F}_0 = 2$, and the solution corresponding to $\mathscr{F}_0 = \mathscr{F}_{\mathscr{K}}(\mu)$ and are represented for $\mu = \frac{4}{3}$.

the differences: the thickness profiles for $\alpha > 1$ are quite similar to those for $\alpha = 1$, while those for $\alpha < 1$ are strikingly different; on the other hand, the velocity profiles for $\alpha > 1$ are very different to those for $\alpha = 1$, while those for $\alpha < 1$ are more similar.

5.3. *Solutions associated to the integral curve* $\mathscr{C}(Z = \frac{1}{4}V^2)$

The solutions represented by pieces of this curve are of interest as limiting cases of the currents discussed in §4, when one considers the transitions between Type II and Type III, or between Type IV and Type III flows (see figures 3*b* and 6*b*).

For any α, the curve $\mathscr{C}(Z = \frac{1}{4}V^2)$ represents a special solution (that we call solution C) of the autonomous equation (23). The piece FP ($V_P = 1, Z_P = \frac{1}{4}$) of $\mathscr{C}$ represents a continuous solution for $\beta = 2$, for which

$$\eta = \xi/\xi_f = f(V)/V, \quad f(V) = \left(\frac{1-2\mu/3}{1-2\mu/3V}\right)^{1-\delta}. \tag{51}$$

Using (51) one finds the bchaviour of the physical variables:

$$h \sim \xi^2 Z \sim \tfrac{1}{4}f(V)^2, \quad u \sim \xi V \sim f(V). \tag{52}$$

As a function of $\eta, f(V(\eta))$ is monotonically decreasing if $0 < \mu < 1$, constant if $\mu = 1$ and monotonically increasing if $1 < \mu < \frac{3}{2}$. It is interesting to evaluate $f(V)$ at the source and at the front. At the front $f(1) = 1$, and h, u are finite. At the source, $f(\infty) = f_\infty = (1-2\mu/3)^{1-\delta}$. Then, if $0 < \mu < 1$, $f_\infty > 1$ (in particular, $f_\infty(\mu = 0) = e^{\frac{2}{3}}$), if $\mu = 1, f_\infty = 1$, and if $1 < \mu < \frac{3}{2}$ one has $1 > f_\infty > 0$. For $\mu = 1$ this solution coincides with a special case of the uniform flows we discussed above. For $\mu = \frac{3}{2}$ it coincides with the solution represented by the singular point B.

This current corresponds to a supercritical source ($\mathscr{F}_0 = 2$); the transition to the subcritical front region occurs without discontinuity.

There is another special solution related to $\mathscr{C}$. It consists of a piece FC_3 of the integral curve $\mathscr{K}(\mu)$ and a piece C_3P of $\mathscr{C}$. This current is everywhere subcritical except at the point C_3 where it just becomes critical. This solution, which we call K, corresponds to $\mathscr{F}_0 = \mathscr{F}_{\mathscr{K}}(\mu)$.

In figure 12 we represent the depth and velocity profiles of the solutions C, K for $\alpha = \frac{1}{4}$ ($\mu = \frac{4}{3}$). The kink in the profiles of K corresponds to the critical point. These profiles should be compared with those of the solutions for $\beta = 2$ in figures 7, 8 (for example, C with that labelled e in figure 8, and K with d in figure 7) that have the same subcritical part, but are discontinuous and differ in the source part. If one considers a solution like e, figure 8, and takes the limit $\mathscr{F}_0 \to 2$, the position of the jump in the phase plane moves closer to the point C_3 and the magnitude of the discontinuities diminishes, finally disappearing in the limit, when the solution is precisely C. Likewise, if $\mathscr{F}_0 \to \mathscr{F}'_{\mathscr{K}}$ for a solution like d, figure 7, the position of the jump moves closer to the source, which finally is choked when $\mathscr{F}_0 = \mathscr{F}'_{\mathscr{K}}$; then the solution is precisely K.

6. Final remarks and conclusions

Some general comments should be made about the physical interpretation of self-similar solutions, especially when derived from a mathematical formalism like the phase-plane technique (as we do here) and not by starting from a certain initial-value problem and following its evolution (either experimentally, or by numerical simulation) to find its intermediate asymptotic behaviour. It must be kept in mind that self-similarities are exact solutions of 'degenerate' problems (Barenblatt 1979); they are of interest because they reveal the intermediate asymptotics of real non-degenerate problems, when the system has evolved into a regime in which some constant governing parameter has ceased to be relevant. In deriving the similarity solutions by direct construction we lose sight of how to set up an actual experiment to produce such currents. In other words, we get a solution, but it may not be obvious to which initial-value problem it belongs. Often the self-similar solution alone only provides few, if any, clues to this effect. In such a case it is better to be cautious, since until we find some reasonable and physically feasible experiment, or initial-value problem, whose intermediate asymptotics is represented by our self-similar solution, we cannot be fully confident that the latter is physically meaningful. Based on the vast amount of literature on self-similarity (see, for example, Sedov 1959; Stanyukovich 1960; Zel'dovich & Raizer 1968; Gratton 1991, and the references given therein) and the experience of the authors, the self-similarities found by the phase-plane fomalism turn out to be physically meaningful (we do not know exceptions), although sometimes they correspond to quite bizarre initial value-problems (see for example Gratton & Minotti 1990). However, in strict honesty, to give a fully rigorous answer to this question in the present case requires a large amount of additional work, which is clearly beyond the scope of this paper. If one does not pretend to adhere to strict mathematical rigour, we can argue why we believe our solutions are physically realistic.

The self-similar solutions corresponding to a source with increasing inflow ($\alpha > 1$) are regular in the limit $t \to 0$. There is no difficulty in conceiving a real (non-degenerate) problem in which the volume of the current varies according to the law (5) right from the start. Consider, for example a source such as we described in §2.1, i.e. a liquid reservoir that drains through a slit opening at the bottom of one of its sides. It is certainly possible (in principle, at least) to begin the experiment with the slit closed, and, starting at $t = 0$, to gradually increase the width of the slit and simultaneously vary the depth of the liquid in the reservoir, in such a way that the source conditions (6) and (7) are satisfied right from the start. In this idealized experiment, we expect that our self-similar solutions will describe correctly the current, as soon as the conditions of validity of shallow-water theory are satisfied. Or one can imagine a transient time interval, $0 < t < t_0$, during which u and h may vary in some arbitrary way, until at $t = t_0$ the experimenter adjusts his gadgets and from there on the conditions (6) and (7) are fulfilled. In this case we expect that for large times, $t \gg t_0$ (more precisely, well after the C_+ characteristic, satisfying $(\mathrm{d}x/\mathrm{d}t)_{C_+} = u+h^{\frac{1}{2}}$ emanating from the source at $t = t_0$ has overtaken the front of the current) the effect of the initial transient will be negligible and the flow will attain the same self-similar asymptotics as before. We then see that a range of reasonable initial-value problems can be set up, whose intermediate asymptotics is described by these solutions. On physical grounds, we expect that in these cases there will be different types of currents, depending only on the character (discontinuous or not) of the matching between the source-determined and the front-determined parts of the current. The matching depends on the extent of the region of influence of the front, which for β (or $\mathscr{F}_0$) sufficiently small encompasses the whole current and chokes the source leading to Type I currents. When the source is not choked, β and $\mathscr{F}_0$ can be chosen independently (Types II and III). This is precisely what one finds. Similar comments can be made for the cases of sources with constant outflow ($\alpha = 1$) and of the spreading of a constant volume ($\alpha = 0$).

The currents produced by a source with decreasing inflow ($\alpha < 1$) are a different matter, because in a real case we cannot assume that the current follows (6) and (7) right from the start. In fact, according to (6), (7), $u(x = 0)$ and $h(x = 0)$ diverge in the limit $t \to 0$, and the self-similar solutions are singular. However, it is still possible to devise an adequate non-degenerate initial-value problem whose intermediate asymptotics is described by our solutions, but one must imagine that before the source begins to behave according to (5) there is already a current flowing (i.e. the source has been turned on previously). This pre-existing current will have a volume $\mathscr{Q}_0$, certain velocity and depth profiles (which depend on the previous history), and of course will not be self-similar. Then at, say, $t = t_0$ we begin turning off the source so that (6) and (7) are satisfied for $t > t_0$. Our expectation is that the self-similar currents we have found will describe the intermediate asymptotics of this problem, for $t \gg t_0$ (i.e. well after the C_+ charateristic emanating from the source at $t = t_0$ has overtaken the front of the current, and when $\mathscr{Q}(t) \gg \mathscr{Q}_0$, and the details of the profiles at $t = t_0$ have ceased to be relevant). This is why we believe that our self-similar solutions are physically meaningful. However, the fact that there must be a pre-existing current before we turn on the source conditions has some consequences: it is conceivable that the matching between the source-determined and the front-determined parts of the current may now be more complicated than the $\alpha > 1$ case. As said above, the self-similar solution itself only offers faint clues pointing to its parent non-degenerate initial-value problem: without additional knowledge we can only guess some of its traits. Scrutinizing closely the solutions for $\alpha < 1$ one finds, in fact, that in the transition to the intermediate asymptotic regime most of the details of the pre-existing flow lose relevance (as expected), but some mark of its existence remains and shows up in the self-similar solution. Consider, for example, the intermediate region of Type IV currents, where the critical transition takes place. Mathematically, this region is needed as a bridge linking the source and front parts of the current (that cannot be directly matched as in Type II and III currents), but the physical explanation of its cause is not obvious. The intermediate region is indeed a very peculiar feature: it is represented by the unique integral curve $\mathscr{K}$, so that its flow pattern does not depend on $\mathscr{F}_0$, nor on β (which only determine the position of the jumps on $\mathscr{K}$). In other words, the intermediate flow does not depend on the boundary conditions (excepting α). Then it is reasonable to infer that it is a feature left over by the pre-existing flow. To test this conjecture we studied with the method of characteristics an initial-value problem consisting of a current with a critical transition such as those that occur for $\alpha = 1$ (see §5.2.2), which we guess may lead to a Type IV current if we turn off the source as described above (this is suggested by the comparison of the profiles of figures 7 and 11). We find that for appropriate values of $\mathscr{F}_0$ a jump develops between the source and the intermediate region where the critical transition occurs, thus supporting the conjecture, but unfortunately we could not go further since the analysis becomes too involved. We omit details for brevity.

In conclusion, we have studied systematically self-similar gravity currents with variable inflow for plane symmetry. We find that to remove ambiguities (the 'not unique' solutions of GR) it is important to characterize adequately the source by giving its Froude number $\mathscr{F}_0$. To obtain the complete family of self-similarities it is also essential to consider solutions with hydraulic jumps. Then one finds solutions for any (compatible) combination of source and front boundary conditions. The different solutions and the parameter ranges in which they occur are summarized in table 2; italics denote the solutions not described by GR, see also figures 3(*b*), 6(*b*) and 10.

The main results of the present work are the following:

(*a*) There are four types of self-similar currents: continuous solutions (Type I), continuous solutions with a supercritical–subcritical transition (Type II), discontinuous solutions (Type III), and discontinuous solutions having hydraulic jumps and a subcritical–supercritical transition (Type IV). The current is always subcritical near the front, but near the source it is subcritical for Type I currents and supercritical for Types II, III and IV (and for Type I when $\alpha = 1$, $\beta > 1$). Type I solutions were previously known (GR), but Type II, III and IV solutions are novel.

(*b*) Type I currents exist only in some β-intervals (except for $\alpha = 1$, when any β is possible), and for them $\mathscr{F}_0 = \mathscr{F}_0(\beta)$, as found by GR. That $\mathscr{F}_0$ and β cannot be chosen independently for these solutions is a consequence of the source and front regions being causally related. There is a close mathematical relationship between Type I and III currents: starting from a Type III current and taking the limit in which the hydraulic jump moves closer to the source until it finally chokes it, one obtains a Type I current.

(*c*) Type II currents exist for $\alpha > 1$, $\beta > 2$, and $1 < \mathscr{F}_0 < 2$. Their characteristic is that the transition between the supercritical flow in the source region and the subcritical flow near the front occurs without discontinuity, Since the source region is outside the domain of influence of the front, β does not determine $\mathscr{F}_0$, and it is possible to specify arbitrarily (but within a certain range) this property of the source.

(*d*) Type III currents are possible for any α, β provided $\mathscr{F}_0$ is sufficiently large ($\mathscr{F}_0 > \mathscr{F}_{inf}$, where $\mathscr{F}_{inf}$ is a certain lower bound, always $\geqslant 1$), and provided we are not in the $(\alpha, \beta, \mathscr{F}_0)$ range corresponding to Types II and IV. Unlike Type II, Type III solutions have a discontinuous supercritical–subcritical transition. As for Type II, in Type III currents the source region is outside the domain of influence of the front, and this is why β does not determine $\mathscr{F}_0$.

(*e*) Type IV currents occur for $\alpha < 1$, $\beta > 2$, $1 < \mathscr{F}_0 < 2$. There is a three-step transition from the supercritical source flow to the subcritical front flow: first, a hydraulic jump connects the source part of the current with the intermediate subcritical flow; second, this flow has a continuous transition into an intermediate supercritical region; finally, there is a second hydraulic jump connecting the intermediate region and the front region. As for Type II and III currents, the source region is outside the domain of influence of the front, and β does not determine $\mathscr{F}_0$.

(*f*) The spreading of a constant volume of fluid ($\alpha = 0$) leads to Type I solutions only, since there is no source in this case. There are solutions for any finite β, because in addition to the currents with $\beta < 2$ already known from the literature (Fannelop & Waldman 1972; Hoult 1972; GR), we find solutions for $\beta \geqslant 2$. These new solutions have the peculiarity that there is a 'dry' region near the origin of spreading ($x = 0$).

(*g*) For steady inflow ($\alpha = 1$), in addition to the uniform currents of GR, we find for any β solutions of Types II and III, provided $\mathscr{F}_0 > 1$.

α	$\mathscr{F}_0$	β			
		$0<\beta<2$		$2\leqslant\beta<\infty$	$\beta=\infty$
0	—	Type I		*Type I, dry origin*	none
$0<\alpha<1$	$\mathscr{F}_0(\beta)$	Type I		—	—
	$\mathscr{F}_{inf}(\beta)$	Type I		—	—
	$\mathscr{F}_{inf}(\beta)<\mathscr{F}_0$	*Type III*		—	—
	$\mathscr{F}_{\mathscr{X}'}<\mathscr{F}_0<2$	—		*Type IV*	none
	$2<\mathscr{F}_0$	—		*Type III*	none
		$0<\beta<1$		$1<\beta<\infty$	
1	β	Type I		—	—
	$\mathscr{F}_{inf}(\beta)$	Type I		—	—
	$\mathscr{F}_{inf}(\beta)<\mathscr{F}_0$	*Type III*		—	—
	$\mathscr{F}_0<\beta$	—		*Type II*	*Type II*
	$\mathscr{F}_0>\beta$	—		*Type III*	*Type III*
		$0<\beta<\beta_c$	$\beta_c<\beta<2$	$2<\beta<\infty$	
$1<\alpha<4$	$\mathscr{F}_0(\beta)$	Type I	—	—	—
	$\mathscr{F}_{inf}(\beta)$	Type I	—	—	—
	$\mathscr{F}_{inf}(\beta)<\mathscr{F}_0$	*Type III*	—	—	—
	$1<\mathscr{F}_0<2$	—	*Type III*	*Type II*	*Type II*
	$2<\mathscr{F}_0$	—	*Type III*	*Type III*	*Type III*

TABLE 2. Summary of the solutions and their parameter ranges; italics denote solutions not found by GR

Summarizing, for any β there is a family of self-similar solutions that represents currents produced by sources with different combinations of α and $\mathscr{F}_0$. Only when the current is everywhere subcritical (Type I, $\mathscr{F}_0<1$), must β and $\mathscr{F}_0$ be compatible (so that $\mathscr{F}_0=\mathscr{F}_0(\beta)$). In the other cases (Types II, III, IV) it is possible to choose β and $\mathscr{F}_0$ ($\mathscr{F}_0>\mathscr{F}_{inf}, 1$) independently.

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